

Perfect fluid theory and its extensions

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TOPICAL REVIEW

Perfect fluid theory and its extensions

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Abstract

We review the canonical theory for perfect fluids in Euler's and Lagrange's formulations. The theory is related to a description of extended structures in higher dimensions. Internal symmetry and supersymmetry degrees of freedom are incorporated. Additional miscellaneous subjects that are covered include physical topics concerning quantization as well as mathematical issues of volume preserving diffeomorphisms and representations of Chern–Simons terms (= vortex or magnetic helicity).

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1. Introduction

The dynamics of fluids is described by a classical field theory, whose origins lie in the nineteenth century, like Maxwell electrodynamics with which it shares some antecedents. The electromagnetic theory has enjoyed much development: it passed into quantum physics, and the resulting quantum electrodynamics served as a model for its non-Abelian generalization, Yang–Mills theory, which today is at the centre of a quantum field theoretic description for fundamental physics. We believe that fluid dynamics can undergo a similar evolution and play a similar generative role in physics.

Modern (quantum) field theory has expanded concepts and calculational possibilities beyond what was familiar to (classical) field theorists. One learned about higher and unexpected symmetries, which also facilitate partial or complete integrability of the relevant differential equations. Topological and geometric concepts and structures, such as solitons and instantons, were recognized as encoding crucial dynamical information about the models. New entities such as Pontryagin densities and Chern–Simons terms entered into the description of kinematics and dynamics. Degrees of freedom were enlarged and unified by new organizing

principles based on non-Abelian symmetries and supersymmetries. Indeed application of field theory to particle physics has now evolved to a study of extended structures such as strings and membranes, whose mathematical description bears some similarity to the theory of fluids.

The novelties introduced in particle field theories can also be developed for the non-particle field theory of fluid dynamics. Correspondingly, fluid dynamics can illuminate some aspects of particle physics, especially as concerns the extended structures that these days are the focus of attention for particle physicists.

We begin by addressing the question of why fluid mechanics would be interesting in its own right. Fluid mechanics, for most physical situations, is obtained from an underlying particle description by suitable averages of Boltzmann-type equations. Recall that the Boltzmann equation for the single-particle distribution function $f(\mathbf{X}, \mathbf{P}, t)$ is given by

$$\frac{\partial f}{\partial t} + \frac{\mathbf{P}}{m} \cdot \frac{\partial f}{\partial \mathbf{X}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{P}} = \mathcal{C}(f) \quad (1.1)$$

where \mathbf{X}, \mathbf{P} , refer to the coordinates and momenta, the phase space variables, of a single particle of mass m ; \mathbf{F} is the force acting on the particle. $\mathcal{C}(f)$ is the collision integral, which takes into account particle interactions. In the special case of the collisionless limit, i.e., with $\mathcal{C} = 0$ for all f , the Boltzmann equation (1.1) determines the distribution function for single particles obeying the standard classical equations of motion. Solving the Boltzmann equation is not easy. The equilibrium distribution function, $f^{(0)} = n_p$, is a solution of the equation; in particular, $\mathcal{C}(f) = 0$ for the equilibrium solution. The general strategy which has been used for solving equation (1.1) is to seek a perturbative solution of the form $f = f^{(0)} + f^{(1)} + \dots$, with n_p appropriately chosen for bosons and fermions. Transport coefficients and fluid equations of motion can then be obtained from the perturbative corrections.

This approach has the virtue of simplicity and does capture many of the general features of the problem of deriving coarse-grained dynamics, fluid mechanics in particular, from the underlying particle dynamics. However, there are two limitations: first because we are only taking care of the single-particle distributions, and second because the treatment is essentially classical.

Regarding the first point, one could indeed try to be more general by starting with the completely general N -particle Liouville equation for the phase space distribution $\rho(\mathbf{X}_n, \mathbf{P}_n)$, $n = 1, 2, \dots, N$. The one-particle distribution function is then given by $\int d\mu_{N-1} \rho(\mathbf{X}_n, \mathbf{P}_n)$, the two-particle distribution is given by $\int d\mu_{N-2} \rho(\mathbf{X}_n, \mathbf{P}_n)$, etc, where $d\mu_N$ denotes the phase space volume for N particles. The Liouville equation then leads to a hierarchy of kinetic equations, the so-called BBGKY (Bogolyubov–Born–Green–Kirkwood–Yvon) hierarchy, involving higher and higher correlated n -particle distribution functions. (For the one-particle distribution function, we get the Boltzmann equation, but with the collision integral given in terms of the two-particle distribution function.) To be able to solve this infinite hierarchy of equations, one needs to truncate it, very often at just the single-particle distribution function. Therefore, even though a more general formulation is possible, the feasibility of solving these equations limits the kinetic approach to dilute systems near equilibrium, where the truncation can be justified.

Regarding the question of quantum corrections, the needed formalism is that of the Schwinger–Dyson equations with a time contour, the so-called Schwinger–Bakshi–Mahanthappa–Keldysh approach [1]. Green's functions are defined by the generating functional

$$Z[\eta] = \frac{\text{Tr } \rho_0 \mathcal{T}_C \exp(iI_C + i\phi \cdot \eta)}{\text{Tr } \rho_0}, \quad (1.2)$$

where the time integral in the action I_C goes along the contour C from $-\infty$ to ∞ , folds back and goes from ∞ to $-\infty$; \mathcal{T}_C denotes ordering along this time contour. ϕ_μ generically

represents fields of interest, η^μ is a source function and ρ_0 is the thermal density matrix. One can represent $Z[\eta]$ as a functional integral:

$$Z[\eta] = \int d\mu[\phi] \exp(iI_C(\phi) + i\phi \cdot \eta). \quad (1.3)$$

The action is again defined on the time contour. Green's functions for which some of the fields are on the forward time line and some are on the reverse time line will represent the effect of statistical distributions. One has to solve the hierarchy of coupled Schwinger–Dyson equations which follow from (1.3), again truncating them at a certain level to get a description of nonequilibrium phenomena. In practice, one has to carry out semiclassical expansions to simplify these to the point where a solution can be found and again we have a formalism of limited validity.

This discussion shows that we should expect that the regime of validity of the fluid dynamical equations derived within kinetic theory or within the time-contour approach is rather limited, basically a semiclassical regime for dilute systems not too far from equilibrium. However, fluid dynamical equations can also be derived from very general principles, showing that they have a much wider regime of validity, and, indeed in practice, we apply them over such a wider range. This is the ‘universality’ of fluid dynamics. It is this property which shows the value of a study of fluid mechanics in its own right, rather than its derivation from an underlying description in some approximation. Consequently a reconsideration of the whole setting and development of the ideas of fluid mechanics in the context of modern concepts in particle physics is entirely appropriate.

In this introduction, we shall review Lagrange's and Euler's description of fluid kinematics and dynamics, and we shall describe the mapping between the two. We shall discuss the Hamiltonian (canonical) formulation, together with the associated Poisson brackets. We shall also present (configuration space) Lagrangians for fluid motion; in the Eulerian case a Clebsch parametrization is needed. Both nonrelativistic and relativistic systems in various spatial dimensions are treated. These are standard topics and they are well known [2]. However, in our presentation of this familiar material we shall approach the subject with an eye towards the various enhancements of fluid mechanics, which comprise our current research and which are described in the remainder of this review.

1.1. Lagrange and Euler descriptions of a fluid and the relationship between them

The Lagrange description of a fluid focuses on the coordinates of the individual fluid particles. These satisfy a Newtonian equation of motion (in the nonrelativistic case). On the other hand, in Euler's formulation, the fluid is described by a density ρ and velocity \mathbf{v} , which are linked by a continuity equation, while Euler's equation describes the dynamics. Euler's method is akin to a classical field theory in physical spacetime.

In fact one may exemplify the two approaches, and the relation between them, already for a single particle, carrying mass m and located on the coordinate $\mathbf{X}(t)$, whose time evolution is governed by a force \mathbf{F} .

$$\ddot{\mathbf{X}}(t) = \frac{1}{m} \mathbf{F}(\mathbf{X}(t)). \quad (1.4)$$

(Overdot denotes differentiation with respect to explicit time dependence.) This is the ‘Lagrange’ description (of course it is Newton's). Next, we introduce the Eulerian (single-) particle density by

$$\rho(t, \mathbf{r}) = m\delta(\mathbf{X}(t) - \mathbf{r}). \quad (1.5)$$

The δ -function follows the dimensionality of space so that the volume integral of ρ is m . Differentiation (1.5) with respect to t leaves

$$\begin{aligned}\dot{\rho}(t, \mathbf{r}) &= m \frac{\partial}{\partial X^i} \delta(\mathbf{X}(t) - \mathbf{r}) \dot{X}^i(t) \\ &= -\frac{\partial}{\partial r^i} [\dot{X}^i(t) m \delta(\mathbf{X}(t) - \mathbf{r})] = -\frac{\partial}{\partial r^i} [v^i(t, \mathbf{r}) \rho(t, \mathbf{r})],\end{aligned}\quad (1.6)$$

where the Eulerian velocity \mathbf{v} is given by

$$\mathbf{v}(t, \mathbf{r}) = \dot{\mathbf{X}}(t) \quad \text{with} \quad \mathbf{r} = \mathbf{X}(t). \quad (1.7)$$

Note that the velocity function $\mathbf{v}(t, \mathbf{r})$ is only defined at the point $\mathbf{r} = \mathbf{X}(t)$; its value at other points being undetermined and irrelevant. Evidently, the continuity equation is satisfied as a consequence of the above definitions:

$$\dot{\rho}(t, \mathbf{r}) + \nabla \cdot \mathbf{j}(t, \mathbf{r}) = 0 \quad (1.8)$$

$$\mathbf{j}(t, \mathbf{r}) = \mathbf{v}(t, \mathbf{r}) \rho(t, \mathbf{r}) = \dot{\mathbf{X}}(t) m \delta(\mathbf{X}(t) - \mathbf{r}). \quad (1.9)$$

To arrive at the dynamical Euler equation, we differentiate \mathbf{j} with respect to time. From (1.9) it follows that

$$\rho(t, \mathbf{r}) \dot{\mathbf{v}}(t, \mathbf{r}) + \mathbf{v}(t, \mathbf{r}) \dot{\rho}(t, \mathbf{r}) = \ddot{\mathbf{X}}(t) m \delta(\mathbf{X}(t) - \mathbf{r}) + \dot{\mathbf{X}}(t) m \frac{\partial}{\partial X^j} \delta(\mathbf{X}(t) - \mathbf{r}) \dot{X}^j(t). \quad (1.10a)$$

Use of the continuity equation on the left-hand side and Newton's equation (1.4) on the right leaves

$$\begin{aligned}\rho(t, \mathbf{r}) \dot{\mathbf{v}}(t, \mathbf{r}) - \mathbf{v}(t, \mathbf{r}) \nabla \cdot (\mathbf{v}(t, \mathbf{r}) \rho(t, \mathbf{r})) \\ = \mathbf{F}(\mathbf{X}(t)) \delta(\mathbf{X}(t) - \mathbf{r}) - \frac{\partial}{\partial r^j} (\dot{\mathbf{X}}(t) \dot{X}^j(t) m \delta(\mathbf{X}(t) - \mathbf{r})).\end{aligned}\quad (1.10b)$$

Cancelling common terms gives Euler's equation (for a single-particle 'fluid'):

$$\dot{v}^i(t, \mathbf{r}) + v^j(t, \mathbf{r}) \partial_j v^i(t, \mathbf{r}) = \frac{1}{m} F^i(\mathbf{r}). \quad (1.10c)$$

(∂_j denotes a derivative with respect to the components of a spatial vector, which can be \mathbf{X} or \mathbf{r} , or below, \mathbf{x} . If context does not determine unambiguously which vector is involved, it will be specified explicitly, as in (1.10a,b).) Strictly speaking, (1.10c) only holds at the point $\mathbf{r} = \mathbf{X}$. Suitable continuations of the function $\mathbf{v}(t, \mathbf{r})$ away from the point $\mathbf{r} = \mathbf{X}$ can be found that make (1.10c) hold everywhere. Such continuations represent the velocity field of a fictitious accompanying fluid, of which \mathbf{X} is one particle. For a complete dynamical description an appropriate expression for the force \mathbf{F} still needs to be given.

The same development holds for a collection of N particles: (1.4) is replaced by

$$\ddot{\mathbf{X}}_n(t) = \frac{1}{m_n} \mathbf{F}(\mathbf{X}_1(t), \dots, \mathbf{X}_n(t)). \quad (1.11a)$$

The particle label n ranges from 1 to N . We shall take the particles to be identical. Therefore the mass does not carry the n label, and the force \mathbf{F}_n has a functional dependence on \mathbf{X}_n independent of n , and is symmetric under exchange of the remaining $N - 1$ particle coordinates:

$$\ddot{\mathbf{X}}_n(t) = \frac{1}{m} \mathbf{F}(\mathbf{X}_n(t); \{\mathbf{X}_k(t), k \neq n\}). \quad (1.11b)$$

The Eulerian mass density, velocity and current are defined as

$$\rho(t, \mathbf{r}) = m \sum_{n=1}^N \delta(\mathbf{X}_n(t) - \mathbf{r}), \quad (1.12)$$

$$\mathbf{j}(t, \mathbf{r}) = \mathbf{v}(t, \mathbf{r})\rho(t, \mathbf{r}) = m \sum_{n=1}^N \dot{\mathbf{X}}_n(t) \delta(\mathbf{X}_n(t) - \mathbf{r}). \quad (1.13)$$

Similarly to the single-particle case, the function $\mathbf{v}(t, \mathbf{r})$ is defined only at the points $\mathbf{r} = \mathbf{X}_n(t)$. Evidently, the continuity and Euler equations continue to hold.

For the true fluid formulation, we promote the discrete particle label n to a continuous label \mathbf{x} and the Lagrange coordinate $\mathbf{X}_n(t)$ becomes $\mathbf{X}(t, \mathbf{x})$. Frequently, \mathbf{x} is specified by the statement that it describes the fluid coordinate \mathbf{X} at initial time $t = 0$, i.e.

$$\mathbf{X}(0, \mathbf{x}) = \mathbf{x}. \quad (1.14)$$

Thus \mathbf{x} is the comoving coordinate. The dynamics is again Newtonian:

$$\ddot{\mathbf{X}}(t, \mathbf{x}) = \frac{1}{m} \mathbf{F}(\mathbf{X}(t, \mathbf{x})). \quad (1.15)$$

The density and velocity are now defined by

$$\rho(t, \mathbf{r}) = \rho_0 \int d\mathbf{x} \delta(\mathbf{X}(t, \mathbf{x}) - \mathbf{r}), \quad (1.16)$$

$$\mathbf{j}(t, \mathbf{r}) = \mathbf{v}(t, \mathbf{r})\rho(t, \mathbf{r}) = \rho_0 \int d\mathbf{x} \dot{\mathbf{X}}(t, \mathbf{x}) \delta(\mathbf{X}(t, \mathbf{x}) - \mathbf{r}). \quad (1.17)$$

The integration is over the entire relevant volume, be it one dimensional, two dimensional or three dimensional. (The dimensionality of the measure will be specified only when formulae are dimension specific.) ρ_0 is a background mass density, so that the volume integral of ρ is the total mass. In the above, we assume that the particles that are nearby in space have similar velocities and thus the function of discrete points $\mathbf{v}(t, \mathbf{X}_n)$ goes over to a smooth continuous function $\mathbf{v}(t, \mathbf{r})$. This is the fundamental assumption in classical fluids. The more general case calls for a phase space density (Boltzmann) description of the collection of particles, which takes us away from the realm of classical Lagrange fluids.

The continuity and Euler equations follow as before, with the latter reading

$$\dot{v}^i(t, \mathbf{r}) + v^j(t, \mathbf{r}) \partial_j v^i(t, \mathbf{r}) = \frac{1}{m} F^i(\mathbf{r}). \quad (1.18)$$

The determination of the force term in the above equation requires some discussion. For external forces, $\mathbf{F}(\mathbf{r})$ is simply a known function of space. For inter-particle forces, however, the force could be a nonlocal function depending on the full distribution of particles in space. We shall postulate that interparticle forces are short range and thus depend only on the distribution of nearby particles. For such forces, \mathbf{F} would depend only on the density of particles and its derivatives at the point \mathbf{r} . Under additional assumptions of isotropicity, \mathbf{F} will be proportional to the gradient of the density, $\nabla \rho$. In this case, a standard argument shows that the right-hand side of (1.18) should be set to $-\frac{1}{\rho} \nabla P$, where P is the pressure [2]. Thus once an equation of state is given, i.e. once the dependence of P on ρ is known, we have a self-contained system of equations—the continuity and Euler equations:

$$\dot{\rho}(t, \mathbf{r}) + \nabla \cdot (\mathbf{v}(t, \mathbf{r})\rho(t, \mathbf{r})) = 0 \quad (1.19)$$

$$\dot{\mathbf{v}}(t, \mathbf{r}) + (\mathbf{v}(t, \mathbf{r}) \cdot \nabla) \mathbf{v}(t, \mathbf{r}) = -\frac{1}{\rho} \nabla P(\rho). \quad (1.20)$$

A fluid obeying these equations is called ‘perfect’.

Later this formalism will be generalized to account for an internal symmetry. For that purpose, it will be useful to delineate explicitly the effect of the δ -functions in (1.16) and

(1.17). In the course of the \mathbf{x} integral, \mathbf{x} is evaluated at a function $\chi(t, \mathbf{r})$, which is inverse of $\mathbf{X}(t, \mathbf{r})$:

$$\mathbf{X}(t, \chi(t, \mathbf{r})) = \mathbf{r} \quad (1.21a)$$

$$\chi(t, \mathbf{X}(t, \mathbf{x})) = \mathbf{x}. \quad (1.21b)$$

(Uniqueness is assumed.) Thus the \mathbf{x} integration sets \mathbf{X} equal to \mathbf{r} and also there is a Jacobian, $\det \frac{\partial X^i}{\partial x^j}$ for the $\mathbf{X} \rightarrow r$ transformation. Consequently from (1.16) and (1.17) it follows that

$$\rho = \rho_0 \frac{1}{\left| \det \frac{\partial X^i}{\partial x^j} \right|_{\mathbf{x}=\chi}}, \quad (1.22)$$

$$\mathbf{v} = \dot{\mathbf{X}}|_{\mathbf{x}=\chi}. \quad (1.23)$$

In other words \mathbf{X} effects a diffeomorphism $\mathbf{x} \rightarrow \mathbf{X} \equiv \mathbf{r}$, while χ acts similarly for $\mathbf{r} \rightarrow \mathbf{x}$. The interchange between a dependent variable \mathbf{X} and an independent variable \mathbf{r} is called a hodographic transformation.

1.2. Lagrangian, Hamiltonian formulations and symmetries of dynamics

1.2.1. *Lagrangian and Hamiltonian functions for Lagrange fluid mechanics.* Since the Lagrange method is essentially Newtonian, Lagrangian and Hamiltonian formulations for (1.15) are readily constructed provided the force is derived from a potential \mathcal{V} which depends on \mathbf{X} and its derivatives.

$$L_L = \int dx \left(\frac{1}{2} m \dot{\mathbf{X}}^2 - \mathcal{V} \right) \quad (1.24a)$$

$$\mathbf{F}(\mathbf{X}) = -\nabla \mathcal{V}(\mathbf{X}) \quad (1.24b)$$

$$H_L = \int dx \left(\frac{1}{2m} \mathbf{P}^2 + \mathcal{V} \right) \quad (1.25a)$$

$$\mathbf{P} = m \dot{\mathbf{X}} \quad (1.25b)$$

A canonical formulation for the Lagrange description of fluid motion follows in the usual way, so that bracketing with H_L and using conventional Poisson brackets for \mathbf{X} and \mathbf{P} reproduces the equation of motion (1.15).

1.2.2. *Diffeomorphism symmetry of Lagrange fluid mechanics.* In the discrete antecedent to the continuum formulation there is the obvious freedom of renaming the n label. The continuum version of this freedom manifests itself in that the Lagrange formulation of fluid dynamics enjoys invariance against volume-preserving diffeomorphisms of the continuous label \mathbf{x} .

An infinitesimal diffeomorphism of \mathbf{x} , generated by an infinitesimal function $\mathbf{f}(\mathbf{x})$ reads

$$\delta_f \mathbf{x} = -\mathbf{f}(\mathbf{x}), \quad (1.26a)$$

and this is volume-preserving when \mathbf{f} is transverse:

$$\nabla \cdot \mathbf{f} = 0. \quad (1.26b)$$

Provided the Lagrange coordinate transforms as a scalar

$$\delta_f \mathbf{X}(t, \mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \nabla \mathbf{X}(t, \mathbf{x}), \quad (1.27)$$

L_L is invariant:

$$\begin{aligned}\delta_f L_L &= \int dx \left(m \dot{X}^i f^j \partial_j \dot{X}^i - \frac{\partial}{\partial X^i} \mathcal{V}(\mathbf{X}) f^j \partial_j X^i \right) \\ &= \int dx f^j \partial_j \left(\frac{m}{2} \dot{\mathbf{X}}^2 - \mathcal{V}(\mathbf{X}) \right).\end{aligned}\quad (1.28)$$

The last expression vanishes after partial integration by virtue of (1.26*b*). (Surface terms are ignored.) Noether's theorem then gives the constant of motion associated with the relabelling symmetry (volume-preserving diffeomorphism):

$$C_f = \int dx \dot{X}^i f^j \partial_j X^i. \quad (1.29)$$

Since \mathbf{f} is an arbitrary transverse function it can be stripped away from (1.29). Explicit formulae reflect the spatial dimensionality of the system. In three dimensions we can present the transverse f^i as $\varepsilon^{ijk} \partial_j \tilde{f}^k$, \tilde{f}^k arbitrary, leading to conserved local vector quantities:

$$C_{(3)}^i = \varepsilon^{ijk} \partial_j \dot{X}^\ell \partial_k X^\ell. \quad (1.30)$$

In two dimensions f^i involves a scalar function, $f^i = \varepsilon^{ij} \partial_j f$, and for planar systems the conserved quantity is a local scalar:

$$C_{(2)} = \varepsilon^{ij} \partial_i \dot{X}^k \partial_j X^k. \quad (1.31)$$

Finally, in one dimension, a 'transverse' function is constant, so the constant of motion for lineal systems remains integrated:

$$C_{(1)} = \int dx^1 \dot{X} \partial_{x^1} X = \int dX \dot{X}. \quad (1.32)$$

1.2.3. Hamiltonian function for Euler fluid mechanics. A Hamiltonian for the continuity and Euler equations (1.19) and (1.20) is obtained from H_L (1.25*a*) by transforming to Eulerian variables. First, however, to ensure proper dimensionality the kinetic term is divided by the spatial volume so that m is replaced by ρ_0 . Also to reproduce the pressure form of the forces, we write \mathcal{V} as a function of the Jacobian, $\det \frac{\partial X^i}{\partial x^j}$. Thus H_L reads

$$H_L = \int dx \left(\frac{1}{2} \rho_0 \dot{\mathbf{X}}^2 + \mathcal{V} \left(\left| \det \frac{\partial X^i}{\partial x^j} \right| / \rho_0 \right) \right). \quad (1.33)$$

The transformation to Eulerian variables is effected by multiplying (1.33) by unity, in the form $\int dr \delta(\mathbf{X} - \mathbf{r})$, and interchanging orders of integration. In this way H_L becomes H_E :

$$H_E = \int dr \left(\frac{1}{2} \rho \mathbf{v}^2 + \frac{\rho}{\rho_0} \mathcal{V} \left(\frac{1}{\rho} \right) \right). \quad (1.34)$$

Agreement with the pressure expression for the force, as in (1.20), is achieved when we identify

$$P(\rho) = -\frac{1}{\rho_0} \mathcal{V}'(1/\rho). \quad (1.35)$$

(The dash denotes derivative with respect to argument.) In the subsequent we drop the subscript E on H and rename $\frac{\rho}{\rho_0} \mathcal{V}(1/\rho)$ as $V(\rho)$. Thus the Euler Hamiltonian reads

$$H = \int dr \left(\frac{1}{2} \rho \mathbf{v}^2 + V(\rho) \right), \quad (1.36)$$

and the pressure is the Legendre transform of V :

$$P(\rho) = \rho V'(\rho) - V(\rho). \quad (1.37)$$

Conventional nomenclature for $V'(\rho)$ is enthalpy and $\sqrt{P'(\rho)} = \sqrt{\rho V''(\rho)}$ is the speed of sound s .

Bracketing the Hamiltonian with ρ and \mathbf{v} should generate equations (1.19) and (1.20). To verify this, we need to know the brackets of ρ , \mathbf{v} with each other. These may be obtained from the canonical brackets in the Lagrange formulation, the only non-vanishing one being

$$\{\dot{X}^i(\mathbf{x}), X^j(\mathbf{x}')\} = \frac{1}{\rho_0} \delta^{ij} \delta(\mathbf{x} - \mathbf{x}'). \quad (1.38)$$

Using the definitions of ρ and \mathbf{j} in terms of \mathbf{X} and $\dot{\mathbf{X}}$, equations (1.16) and (1.17), as well as the canonical brackets (1.38), determines the brackets of ρ and \mathbf{j} :

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = 0 \quad (1.39)$$

$$\{j^i(\mathbf{r}), \rho(\mathbf{r}')\} = \rho(\mathbf{r}) \partial_i \delta(\mathbf{r} - \mathbf{r}') \quad (1.40)$$

$$\{j^i(\mathbf{r}), j^j(\mathbf{r}')\} = j^j(\mathbf{r}) \partial_i \delta(\mathbf{r} - \mathbf{r}') + j^i(\mathbf{r}') \partial_j \delta(\mathbf{r} - \mathbf{r}'). \quad (1.41)$$

Since $\mathbf{j} = \mathbf{v}\rho$ these in turn imply that the brackets for ρ and \mathbf{v} take the form [3]

$$\{v^i(\mathbf{r}), \rho(\mathbf{r}')\} = \partial_i \delta(\mathbf{r} - \mathbf{r}'), \quad (1.42)$$

$$\{v^i(\mathbf{r}), v^j(\mathbf{r}')\} = -\frac{\omega_{ij}(\mathbf{r})}{\rho(\mathbf{r})} \delta(\mathbf{r} - \mathbf{r}'), \quad (1.43)$$

where

$$\omega_{ij}(\mathbf{r}) = \partial_i v_j(\mathbf{r}) - \partial_j v_i(\mathbf{r}) \quad (1.44)$$

is called the fluid vorticity. (In the above expressions, (1.38)–(1.44), all quantities are at equal times, so the time argument is omitted.) Of course the Jacobi identity is satisfied by the brackets.

It is now straightforward to verify from (1.39), (1.42)–(1.44) that bracketing with H reproduces the equations of motion (1.19) and (1.20).

$$\dot{\rho} = \{H, \rho\} = -\nabla \cdot (\mathbf{v}\rho) \quad (1.45)$$

$$\dot{\mathbf{v}} = \{H, \mathbf{v}\} = -(\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla V'(\rho). \quad (1.46)$$

These equations may also be presented as continuity equations for a (nonrelativistic) energy–momentum tensor. The energy density

$$\mathcal{E} = \frac{1}{2} \rho \mathbf{v}^2 + V = T^{00}, \quad (1.47)$$

together with the energy flux

$$T^{j0} = \rho v^j \left(\frac{1}{2} \mathbf{v}^2 + V' \right), \quad (1.48)$$

obeys

$$\dot{T}^{00} + \partial_j T^{j0} = 0. \quad (1.49)$$

Similarly the momentum density, \mathcal{P} , which in the nonrelativistic theory coincides with the current,

$$\mathcal{P}^i = \rho v^i = T^{0i}, \quad (1.50a)$$

and the stress tensor T^{ij}

$$T^{ij} = \delta^{ij}(\rho V' - V) + \rho v^i v^j = \delta^{ij} P + \rho v^i v^j, \quad (1.50b)$$

satisfy

$$\dot{T}^{0i} + \partial_j T^{ji} = 0. \quad (1.51)$$

Note that $T^{0i} \neq T^{i0}$ because the theory is not Lorentz invariant, but $T^{ij} = T^{ji}$ because it is invariant against spatial rotations. ($T^{\mu\nu}$ is not, properly speaking, a ‘tensor’, but an energy–momentum ‘complex’.) Because $\mathcal{P} = \mathbf{j}$, the current algebra (1.41) is also the momentum density algebra.

1.2.4. Symmetries of Euler fluid mechanics. The above continuity equations and other specific properties of the energy–momentum tensor allow constructing constants of motion, which reflect symmetries of theory. The Hamiltonian [= energy],

$$E = \int dr \mathcal{E} \quad (\text{time translation}), \quad (1.52)$$

is constant as a consequence of time-translation invariance, while the constancy of the momentum,

$$\mathbf{P} = \int dr \mathcal{P} = \int dr \mathbf{j} \quad (\text{space translation}), \quad (1.53)$$

follows from space-translation invariance. The index symmetry of T^{ij} is a consequence of rotational invariance and ensures that the angular momentum,

$$M^{ij} = \int dr (r^i \mathcal{P}^j - r^j \mathcal{P}^i) \quad (\text{spatial rotation}), \quad (1.54)$$

is constant. The identity of the momentum density and the current density allows construction of the Galileo boost constant of motion,

$$\mathbf{B} = t\mathbf{P} - \int dr \mathbf{r} \rho \quad (\text{velocity boost}), \quad (1.55)$$

whose time independence signals invariance against velocity boosts. Finally, the total number

$$N = \int dr \rho \quad (\text{number}), \quad (1.56)$$

is also conserved, as a consequence of the continuity equation (1.19). Upon bracketing with each other, the constants (1.52)–(1.56) form the extended Galileo Lie algebra, and they generate Galileo transformations on ρ and \mathbf{v} . Brackets with N vanish; N provides the central extension for the Galileo algebra in the $\{\mathbf{B}, \mathbf{P}\}$ bracket.

Further constants of motion are present for specific forms of V . In particular if

$$2T^{00} = \delta_{ij} T^{ij}, \quad (1.57a)$$

which in d spatial dimensions requires that

$$V(\rho) = \lambda \rho^{1+\frac{2}{d}}, \quad (1.57b)$$

two more constants exist. They are the dilation,

$$D = 2tH - \int dr \mathbf{r} \cdot \mathcal{P} \quad (\text{dilation}), \quad (1.58)$$

and the special conformal generator,

$$K = t^2 H - tD - \frac{1}{2} \int dr \mathbf{r}^2 \rho \quad (\text{conformal transformation}). \quad (1.59)$$

The latter two, together with H , form the $SO(2, 1)$ Lie algebra of the nonrelativistic conformal group, which they generate by bracketing. Together with the remaining Galileo group elements this is called the Schrödinger group [4]⁴.

Finally, in any dimension when

$$V(\rho) = \lambda/\rho, \tag{1.60}$$

we are dealing with a Chaplygin gas and, as we shall discuss below, this model supports remarkable higher symmetries related to relativistic extended objects [5].

What is the Eulerian image for the volume-preserving diffeomorphism symmetries of the Lagrange formulation discussed in section 1.2.2? First we note that the Eulerian variables ρ and \mathbf{v} do not respond to the volume-preserving transformations on the Lagrange variable \mathbf{X} . This is seen from (1.16), (1.17), (1.27). Also the associated constants of motion, summarized by (1.29)–(1.32) do not possess in general simple Eulerian counterparts. But some do.

Let us begin with the three-dimensional quantity in (1.30) and integrate it over a surface $\bar{\mathbf{S}}$ in the \mathbf{X} parameter space bounded by a closed curve $\partial\bar{\mathbf{S}}$. This gives the constant

$$A = \int_{\bar{\mathbf{S}}} dS^i \varepsilon^{ijk} \partial_j (\partial_k X^\ell \dot{X}^\ell) = \oint_{\partial\bar{\mathbf{S}}} dX^i \partial_i X^\ell \dot{X}^\ell. \tag{1.61a}$$

Upon performing the diffeomorphism $\mathbf{x} \rightarrow \chi$, equation (1.61a) is expressed in terms of Euler variables as

$$A = \oint_{\partial\bar{\mathbf{S}}} d\mathbf{r} \cdot \mathbf{v} = \int_{\bar{\mathbf{S}}} d\mathbf{S} \cdot \boldsymbol{\omega}, \tag{1.61b}$$

where $\partial\bar{\mathbf{S}}$ is the (time-dependent) image of $\partial\mathbf{S}$ under the diffeomorphism. The above quantity, known as the velocity circulation or vorticity flux, is therefore constant; this is Kelvin’s theorem. (One can establish the result directly for (1.61b) from the equations of motion for Euler variables, provided one takes into account the time dependence of the contour $\partial\mathbf{S}$.) Similarly, conserved velocity circulation exists also in two-dimensional fluid mechanics; of course the integration surface or contour then lie in the plane.

Additional important constants arise from the volume integrals of the $C_{(d)}^i$, $d = 1, 2, 3$, in (1.30)–(1.32). In three dimensions, we begin with (1.30) contracted with $\dot{X}^m \partial_i X^m$ and integrated over volume:

$$\begin{aligned} C_{(3)} &= \int d^3x \varepsilon^{ijk} (\dot{X}^m \partial_i X^m) \partial_j (\dot{X}^n \partial_k X^n) \\ &= \int d^3x \dot{X}^m \partial_j \dot{X}^n \varepsilon^{ijk} \partial_i X^m \partial_k X^n. \end{aligned} \tag{1.62a}$$

We use the identity $\varepsilon^{ijk} \partial_i X^m \partial_k X^n = \varepsilon^{m\ell n} \frac{\partial x^j}{\partial X^\ell} \det \frac{\partial X^p}{\partial x^q}$, and multiply $C_{(3)}$ by unity in the form $\int d^3r \delta(\mathbf{X} - \mathbf{r})$ so that $C_{(3)}$ becomes

$$\begin{aligned} C_{(3)} &= \int d^3r d^3x \dot{X}^m \frac{\partial \dot{X}^n}{\partial x^j} \varepsilon^{m\ell n} \frac{\partial x^j}{\partial X^\ell} \det \frac{\partial X^p}{\partial x^q} \delta(\mathbf{X} - \mathbf{r}) \\ &= \int d^3r \mathbf{v} \cdot (\nabla \times \mathbf{v}) = \int d^3r \mathbf{v} \cdot \boldsymbol{\omega}, \end{aligned} \tag{1.62b}$$

(apart from an irrelevant factor of $\pm\rho_0$). Conservation of $C_{(3)}$, called the vortex helicity, also follows when the Euler equation (1.20) is applied to (1.62b).

⁴ $SO(2, 1)$ together with the Galileo group, equivalently the ‘Schrödinger’ group, provides the maximal symmetry group for the kinetic term of nonrelativistic dynamics. Various interactions in various dynamical systems preserve the $SO(2, 1)$ symmetry.

An expression like $C_{(3)}$, with \mathbf{v} an arbitrary 3-vector, is a well-known mathematical entity, called the Chern–Simons term. In the last 20 years, Chern–Simons terms have come to play a significant role in physics and mathematics [6]. We shall have more to say about them, incarnated in various contexts, but always in odd-dimensional space. Indeed the one-dimensional constant (1.32), when written in terms of Euler variables reads

$$C_{(1)} = \int d^1 r v. \quad (1.63)$$

(Here ‘ r ’ is not a radial coordinate, but lies on the real line.) This is just a one-dimensional Chern–Simons term. (We shall further discuss Chern–Simons terms in appendix B.)

Finally we turn to the planar constant $C_{(2)}$ (1.31). Taking it to the M th power, multiplying by $\delta(\mathbf{X} - \mathbf{r})$ and integrating over \mathbf{x} and \mathbf{r} gives

$$C_{(2)}^M = \int d^2 x d^2 r (\varepsilon^{ij} \partial_i \dot{X}^k \partial_j X^k)^M \delta(\mathbf{X} - \mathbf{r}). \quad (1.64a)$$

The 2×2 matrix identity $\varepsilon^{ij} \frac{\partial X^k}{\partial x^j} = \varepsilon^{mk} \frac{\partial x^i}{\partial X^m} \det \frac{\partial X^p}{\partial x^q}$ converts the above to

$$\begin{aligned} C_{(2)}^M &= \int d^2 x d^2 r \left(\frac{\partial \dot{X}^k}{\partial x^i} \varepsilon^{mk} \frac{\partial x^i}{\partial X^m} \det \frac{\partial X^p}{\partial x^q} \right)^M \delta(\mathbf{X} - \mathbf{r}) \\ &= \int d^2 r \rho \left(\frac{\partial}{\partial r^m} v^k \varepsilon^{mk} / \rho \right)^M = \int d^2 r \rho \left(\frac{\omega}{\rho} \right)^M \end{aligned} \quad (1.64b)$$

apart from irrelevant factors. Here ω is the planar vorticity $\varepsilon^{ij} \partial_i v^j$. Thus in the plane there is a denumerably infinite set of particle relabelling constants. Again, the time independence of $C_{(2)}^M$ can be established directly from (1.64b) with the help of the continuity and Euler equations (1.19) and (1.20).

The brackets of the above volume-integrated relabelling constants with the Euler Hamiltonian vanish, because they are time independent. But this does not depend on the specific form of the Hamiltonian, since the fundamental brackets (1.39)–(1.44) give vanishing brackets for $C_{(d)}$ with ρ and \mathbf{v} . This is as it should be, because we have already remarked that the Eulerian variables do not respond to the relabelling transformations. Quantities whose bracket vanishes with all the elements of a (bracket) algebra, here ρ and \mathbf{v} , are called Casimir invariants. So we see that the relabelling symmetry gives rise to Casimir invariants in the Euler formulation for fluids. This has a profound impact on the possibility of constructing a Lagrangian for Eulerian fluids.

Finally, let us remark that even though the volume-preserving diffeomorphism transformations do not act on the Eulerian ρ and \mathbf{v} , there remains in the formalism a related structure: the brackets (1.41) of currents (equivalently, momentum densities) present a local realization of the full (not merely volume-preserving) diffeomorphism algebra. Since if we define for arbitrary \mathbf{f}

$$j_{\mathbf{f}} = \int d\mathbf{r} \mathbf{f}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}), \quad (1.65)$$

then (1.41) implies

$$\{j_{\mathbf{f}_1}, j_{\mathbf{f}_2}\} = j_{\mathbf{f}_{12}}, \quad (1.66)$$

where \mathbf{f}_{12} is the Lie bracket of \mathbf{f}_1 and \mathbf{f}_2

$$f_{12}^i = f_1^j \partial_j f_2^i - f_2^j \partial_j f_1^i. \quad (1.67)$$

1.2.5. *Lagrange function for Eulerian fluid mechanics.* While constructing the Euler Hamiltonian is straightforward, for example by transforming the Lagrange Hamiltonian, as in (1.33)–(1.36), an analogous construction for the Euler Lagrangian is problematic. First, the equations for the variables ρ and \mathbf{v} are first order in time, so the Lagrangian should reproduce this. Second, time derivatives in the Lagrangian determine the canonical bracket structure, which ultimately should reproduce (1.39)–(1.44). However, direct transcription of L_L (1.24a) to L_E , analogous to the passage from H_L in (1.25a) to H_E (1.36), would yield $L_E = \int dr (\frac{1}{2}\rho\mathbf{v}^2 - V(\rho))$, which contains no time derivatives. So something else must be done. Moreover, as we shall now explain, the presence of the Casimirs $C_{(d)}$ and N poses obstructions to the construction of a Lagrangian, which must be overcome.

We present the relation between Lagrangians and the canonical bracket structure in appendix A.

To construct a Lagrangian for Euler fluids, which leads to the correct equations of motion (1.19) and (1.20) and brackets (1.39)–(1.44), we must neutralize the Casimirs. In three and one spatial dimensions, we must neutralize the velocity Chern–Simons terms (1.62) and (1.63), and also the total number N . In two dimensions the Casimirs which must be neutralized comprise the infinite tower (1.64), as well as N .

In three dimensions this is achieved in the following manner, based on an idea of Lin [10]. We use the Clebsch parametrization for the vector field \mathbf{v} [11]. Any three-dimensional vector, which involves three functions, can be presented as

$$\mathbf{v} = \nabla\theta + \alpha\nabla\beta, \tag{1.68}$$

with three suitably chosen scalar functions θ, α and β . This is called the Clebsch parametrization, and (θ, α, β) are called Monge potentials. In this parametrization, the vorticity reads

$$\boldsymbol{\omega} = \nabla\alpha \times \nabla\beta, \tag{1.69}$$

and the Lagrangian is taken as

$$L = - \int d^3 r \rho (\dot{\theta} + \alpha\dot{\beta}) - H|_{\mathbf{v}=\nabla\theta+\alpha\nabla\beta}, \tag{1.70}$$

with \mathbf{v} in H expressed as in (1.68). Thus the canonically conjugate pairs are (ρ, θ) and $(\rho\alpha, \beta)$ replacing ρ and \mathbf{v} . The phase space $(\rho, \theta, \alpha, \beta)$ is four dimensional, corresponding to the four observables ρ and \mathbf{v} , and a straightforward calculation shows that the Poisson brackets (1.39), (1.42)–(1.44) are reproduced with \mathbf{v} constructed by (1.68).

But how has the obstacle presented by the Casimirs been overcome? Let us observe that in the Clebsch parametrization C_3 is given by

$$C_3 = \int d^3 r \epsilon^{ijk} \partial_i\theta \partial_j\alpha \partial_k\beta, \tag{1.71}$$

which is just a surface integral

$$C_3 = \int d\mathbf{S} \cdot (\theta\boldsymbol{\omega}). \tag{1.72}$$

In this form, C_3 has no bulk contribution, and presents no obstacle to constructing a symplectic 2-form and a canonical 1-form in terms of $(\rho, \theta, \alpha, \beta)$, which are defined in the bulk, that is, for all finite \mathbf{r} . Moreover, the brackets with N are no longer universally vanishing. Specifically with the new dynamical variable θ we find

$$\{N, \theta\} = -1. \tag{1.73}$$

Note that the response of θ to a finite boost with velocity \mathbf{u}

$$\theta(t, \mathbf{r}) \rightarrow \theta(t, \mathbf{r} - \mathbf{u}t) + \mathbf{u} \cdot \mathbf{r} - \frac{\mathbf{u}^2}{2}t, \quad (1.74)$$

contains the 1-cocycle $\mathbf{u} \cdot \mathbf{r} - \frac{\mathbf{u}^2}{2}t$, as is familiar from representations of the Galileo group.

A ‘derivation’ of (1.70) can be given, based on ideas of Lin and earlier ones of Eckart [12]. We begin with L_E , the Euler transcription of the Lagrange Lagrangian, mentioned earlier, which contains no time derivatives. This is supplemented with constraints, enforced by Lagrange multipliers that ensure various continuity equations:

$$L = \int d^3r \left(\frac{1}{2} \rho \mathbf{v}^2 - V(\rho) + \theta(\dot{\rho} + \nabla \cdot (\mathbf{v}\rho)) - \rho\alpha(\dot{\beta} + \mathbf{v} \cdot \nabla\beta) \right). \quad (1.75)$$

The first two terms in the integrand reproduce L_E ; the first multiplier, θ , enforces the matter/current continuity equation. The second continuity equation for β , enforced by the multiplier $\rho\alpha$, is physically obscure. (Lin argues that it reflects the conservation of ‘initial data’ for fluid motion.) Varying \mathbf{v} and eliminating it gives (1.70).

Let us carry out all the variations and re-derive the two Eulerian equations. For greater generality, useful for relativistic kinematics, we shall take an arbitrary kinetic energy $\rho T(\mathbf{v})$, rather than the above nonrelativistic case $T(\mathbf{v}) = \frac{1}{2}\mathbf{v}^2$, and define that the momentum \mathbf{p} be the derivative of T :

$$\mathbf{p} \equiv \frac{\partial T(\mathbf{v})}{\partial \mathbf{v}}. \quad (1.76)$$

Varying θ gives the continuity equation (1.19); varying α gives

$$\dot{\beta} + \mathbf{v} \cdot \nabla\beta = 0. \quad (1.77a)$$

and varying β gives a similar continuity equation for α .

$$\dot{\alpha} + \mathbf{v} \cdot \nabla\alpha = 0. \quad (1.77b)$$

Next we vary \mathbf{v} to find

$$\rho\mathbf{p} - \rho(\nabla\theta + \alpha\nabla\beta) = 0. \quad (1.78)$$

Thus in the general case it is \mathbf{p} (rather than \mathbf{v}) that is given by the Clebsch parametrization. With (1.78) the Lagrangian (1.75) is rewritten, apart from a total time derivative, as

$$L = \int d^3r (\rho T(\mathbf{v}) - V(\rho) - \rho(\dot{\theta} + \alpha\dot{\beta}) - \rho\mathbf{v} \cdot \mathbf{p}). \quad (1.79)$$

Let $h(\mathbf{p})$ be the Legendre transform of $T(\mathbf{v})$

$$h(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} - T(\mathbf{v}), \quad (1.80a)$$

$$\frac{\partial h(\mathbf{p})}{\partial \mathbf{p}} = \mathbf{v}, \quad (1.80b)$$

and L becomes

$$L = \int d^3r (-\rho(\dot{\theta} + \alpha\dot{\beta}) - \rho h(\mathbf{p}) - V(\rho)). \quad (1.81)$$

The remaining variable to vary is ρ :

$$\dot{\theta} + \alpha\dot{\beta} + h(\mathbf{p}) + V'(\rho) = 0 \quad (1.82a)$$

Differentiating with respect to r^i converts this to

$$\partial_i \dot{\theta} + \partial_i \alpha \dot{\beta} + \alpha \partial_i \dot{\beta} = -\frac{\partial h(\mathbf{p})}{\partial p^j} \partial_i p^j - V''(\rho) \partial_i \rho. \quad (1.82b)$$

But according to (1.78) $\partial_i p^j = \partial_j p^i + \partial_i \alpha \partial_j \beta - \partial_j \alpha \partial_i \beta$. Thus the first term on the right-hand side in (1.82b) is

$$-v^j \partial_j p^i - \partial_i \alpha v^j \partial_j \beta + v^j \partial_j \alpha \partial_i \beta = -v^j \frac{\partial p^i}{\partial v^k} \partial_j v^k + \partial_i \alpha \dot{\beta} - \dot{\alpha} \partial_i \beta,$$

where we have used (1.77) and (1.78). Rearranging (1.82b) leads to

$$\partial_i \dot{\theta} + \dot{\alpha} \partial_i \beta + \alpha \partial_i \dot{\beta} + v^j \partial_j p^i = -V''(\rho) \partial_i \rho, \quad (1.83a)$$

or

$$\dot{p}^i + v^j \partial_j p^i = -V''(\rho) \partial_i \rho. \quad (1.83b)$$

With Newtonian kinematics $p^i = v^i$, and Euler's equation is regained. With arbitrary kinematics $p^i \equiv \frac{\partial T(\mathbf{v})}{\partial v^i}$. Also, since $\delta p^i = \frac{\partial p^i}{\partial v^j} \delta v^j = \tau^{ij} \delta v^j$ where $\tau^{ij} = \frac{\partial^2 T}{\partial v^i \partial v^j}$, the above becomes

$$\tau^{ik} (\dot{v}^k + v^j \partial_j v^k) = -V''(\rho) \partial_i \rho, \quad (1.84a)$$

or

$$\dot{v}^i + v^j \partial_j v^i = -(\tau^{-1})^{ij} V''(\rho) \partial_j \rho, \quad (1.84b)$$

whenever the inverse of τ^{ij} exists. For Newtonian kinematics $\tau^{ij} = \delta^{ij}$.

Note that the free Euler equation, (1.84b) with $V'' = 0$, is insensitive to the form of the kinetic term $T(\mathbf{v})$ (provided τ^{ij} possess an inverse), and can be solved together with the continuity equation (1.19). The general solution in the non-interacting case is presented in terms of posited initial data:

$$\rho(t = 0, \mathbf{r}) = \rho_0(\mathbf{r}) \quad (1.85)$$

$$\mathbf{v}(t = 0, \mathbf{r}) = \mathbf{v}_0(\mathbf{r}). \quad (1.86)$$

Define the quantity $\chi(t, \mathbf{r})$ by the equation

$$\mathbf{r} = t \mathbf{v}_0(\chi(t, \mathbf{r})) + \chi(t, \mathbf{r}). \quad (1.87)$$

Then (1.19) and (1.20) (with $\nabla P = 0$) are solved by

$$\rho(t, \mathbf{r}) = \rho_0(\chi) \left| \det \frac{\partial \chi^i}{\partial r^j} \right|, \quad (1.88)$$

$$\mathbf{v}(t, \mathbf{r}) = \mathbf{v}_0(\chi). \quad (1.89)$$

This result is verified by differentiation; alternatively it may be derived from (1.16), (1.17), with $\mathbf{X}(t, \mathbf{x})$ taken in the absence of forces to be a linear function of t .

In one dimension, we parametrize the velocity as a derivative of a potential θ ,

$$v = \theta', \quad (1.90)$$

and the phase space consists of (ρ, θ) . The Casimir (1.63) again becomes a surface term (only endpoints contribute) and is neutralized in the bulk. The two variables are the conjugate pair (ρ, θ) , which capture the two degrees of freedom (ρ, v) . Of course lineal fluids possess no vorticity, so the velocity bracket (1.43) vanishes, while (1.42) is verified. (In an alternative approach to lineal fluids, we replace $\theta(r)$ by $\frac{1}{2} \int dr' \varepsilon(r - r') v(r')$ and the canonical 1-form

$\int dr \theta \dot{\rho}$ is $-\frac{1}{2} \int dr dr' \rho(r) \varepsilon(r-r') \dot{v}(r')$, where ε is the ± 1 step function. Evidently this leads to a spatially non-local, but otherwise completely satisfactory canonical formulation for fluids on a line.)

Two dimensions present the additional problem that the number of physical variables is three: (ρ, \mathbf{v}) , but an odd number cannot form a symplectic structure. At the same time there is an infinite number of Casimirs, (1.64). One may then conclude, heuristically, that it should be possible to neutralize an infinite number of non-local Casimirs by suppressing one local degree of freedom, thereby decreasing the effective variables from three to two, an even number with which one can build a symplectic structure. But it is not known how to effect this suppression; rather the Lin/Clebsch method is adopted, which increases the degrees of freedom to 4 and the Lagrangian (1.70) is used in two dimensions as well.

Finally note that the Lagrangian in (1.70), apart from a total time derivative, can also be written as

$$\begin{aligned} L &= - \int dr [\rho(\dot{\theta} + \alpha \dot{\beta}) + \rho \mathbf{v} \cdot (\nabla \theta + \alpha \nabla \beta)] + L_E \\ &= - \int dr j^\mu (\partial_\mu \theta + \alpha \partial_\mu \beta) + L_E. \end{aligned} \quad (1.91)$$

Although we are dealing with nonrelativistic dynamics, we have used covariant notation for the canonical 1-form, with $j^\mu = (c\rho, \rho \mathbf{v})$ and $\partial_\mu = (\frac{1}{c} \frac{\partial}{\partial t}, \nabla)$. This formulation becomes our starting point for the relativistic generalization, discussed in section 1.4 below. (That is why we have introduced the velocity of light c in the above definitions; of course it disappears in the nonrelativistic theory, where it has no role.)

1.3. Irrotational fluids

The simplification found for one-dimensional fluids, (1.90), presenting the velocity as a derivative of a velocity potential θ , can be extended to two- and three-dimensional fluids that are irrotational—the vorticity vanishes:

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0 \quad (1.92)$$

$$\mathbf{v} = \nabla \theta. \quad (1.93)$$

The Clebsch parametrization (1.68) holds trivially; the potentials α, β as well as the Casimirs $C_{(d)}$ vanish, and N obeys (1.73). This removes the obstruction to a canonical formalism. The Lagrangian (1.70) becomes

$$L = - \int dr \rho \dot{\theta} - H|_{\mathbf{v}=\nabla \theta}, \quad (1.94)$$

and this can be derived by the Eckart procedure as in (1.75)–(1.81), where now only the continuity equation is enforced by the Lagrange multiplier θ , and the Monge potentials α and β are omitted.

The Euler equation (1.20), with the pressure P expressed in terms of the enthalpy $V'(\rho)$ as in (1.37) may be integrated once to give the Bernoulli equation:

$$\dot{\theta} + \frac{1}{2} (\nabla \theta)^2 = -V'(\rho). \quad (1.95)$$

This also follows by bracketing θ with H in (1.94), since now there is only one non-vanishing and canonical bracket:

$$\{\theta(\mathbf{r}), \rho(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}'). \quad (1.96)$$

Observe that the Bernoulli equation (1.94) allows for the possibility of expressing ρ in terms of $\dot{\theta} + \frac{1}{2}(\nabla\theta)^2$, through the inverse of V' . One can then eliminate ρ in the continuity equation, leaving a single (non-linear) equation for θ .

1.4. Relativistic fluids

We discuss the formalism for describing relativistic fluids; only the Euler approach is treated. Usually the dynamics of a relativistic fluid is presented in terms of the energy–momentum tensor, $\theta^{\mu\nu}$, and the equations of motion are just the conservation equations $\partial_\mu\theta^{\mu\nu} = 0$. This is analogous to the nonrelativistic situation mentioned previously, where the nonrelativistic energy–momentum complex $T^{\mu\nu}$ encapsulates the equations of motion for a nonrelativistic fluid. (We denote the relativistic energy–momentum tensor by $\theta^{\mu\nu}$, to distinguish it from the nonrelativistic $T^{\mu\nu}$ introduced in (1.47)–(1.51). The limiting relation between the two is given below.) But we shall begin with a Lagrange density.

Inspired by the suggestive formula (1.91), we consider

$$\mathcal{L} = -j^\mu a_\mu - f(\sqrt{j^\mu j_\mu}). \tag{1.97}$$

Here j^μ is the current Lorentz vector $j^\mu = (c\rho, \mathbf{j})$.⁵ The a_μ comprise a set of auxiliary variables; in the relativistic analogue of irrotational fluids we take $a_\mu = \partial_\mu\theta$, more generally

$$a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta, \tag{1.98}$$

so that the Chern–Simons density of a_i is a total derivative (compare (1.71) and (1.72)). The function f depends on the Lorentz invariant $j^\mu j_\mu = c^2\rho^2 - \mathbf{j}^2$ and encodes the specific dynamics (equation of state).

The energy–momentum tensor for \mathcal{L} is

$$\theta_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \frac{j_\mu j_\nu}{\sqrt{j^\alpha j_\alpha}} f'(\sqrt{j^\alpha j_\alpha}). \tag{1.99}$$

(One way to derive (1.99) from (1.97) is to embed that expression in an external metric tensor $\eta_{\mu\nu}$, which is then varied; in the variation j^μ and a_μ are taken to be metric independent and $j_\mu = g_{\mu\nu}j^\nu$.) Furthermore, varying j^μ in (1.97) shows that

$$a_\mu = -\frac{j_\mu}{\sqrt{j^\alpha j_\alpha}} f'(\sqrt{j^\alpha j_\alpha}), \tag{1.100}$$

so that (1.99) becomes

$$\theta_{\mu\nu} = -g_{\mu\nu}[nf'(n) - f(n)] + u_\mu u_\nu nf'(n). \tag{1.101}$$

We have introduced the proper velocity u_μ by factoring j_μ , as suggested by Eckart:

$$j_\mu = nu_\mu \quad u^\mu u_\mu = 1. \tag{1.102}$$

One notes that n is proportional to the proper density and $1/n$ is proportional to the specific volume. Equation (1.101) identifies the proper energy density e and the pressure P (which coincides with \mathcal{L}) through the conventional formula (see, for example, [2] or [15]),

$$\theta_{\mu\nu} = -\eta_{\mu\nu}P + u_\mu u_\nu(P + e). \tag{1.103}$$

Therefore, in our case

$$e = f(n) \tag{1.104}$$

⁵ Note that in conventional treatments, as in [15], ρ is a scalar, rather than the time component of a Lorentz vector. This distinction makes no difference in the non-interacting case, but our choice allows the introduction of self-interactions in a Lorentz invariant way by using the scalar $\sqrt{j^\alpha j_\alpha} \equiv n$.

$$P = nf'(n) - f(n). \quad (1.105)$$

The thermodynamic relation involving entropy S reads

$$P d\left(\frac{1}{n}\right) + d\left(\frac{e}{n}\right) \propto dS, \quad (1.106)$$

where the proportionality constant is determined by the temperature. With (1.104) and (1.105) the left-side of (1.106) vanishes and we verify that entropy is constant, that is, we are dealing with an isentropic system, as has been stated at the very beginning.

For the free system, the pressure vanishes, so we choose $f(n) = cn$:

$$\mathcal{L}_0 = -j^\mu a_\mu - c\sqrt{j^\mu j_\mu}. \quad (1.107)$$

Other forms for f give rise to relativistic fluid mechanics with other equations of state.

Taking the divergence of $\theta_{\mu\nu}$ in (1.101) leaves

$$\partial^\mu \theta_{\mu\nu} = -nf''(n)\partial_\nu n + nu^\mu \partial_\mu (u_\nu f'(n)) + \partial_\mu (nu^\mu)u_\nu f'(n). \quad (1.108)$$

The first two terms on the right-hand side are orthogonal to u^ν , the last one is parallel. So the vanishing of the divergence of $\theta_{\mu\nu}$ implies the continuity equation:

$$0 = \partial_\mu (nu^\mu) = \partial_\mu j^\mu. \quad (1.109)$$

The vanishing of the remaining components is equivalent to

$$u^\mu (\partial_\nu u_\mu - \partial_\mu u_\nu) f'(n) + (g_{\nu\mu} - u_\nu u_\mu) \partial^\mu n f''(n) = 0. \quad (1.110)$$

This is the relativistic Euler equation.

The same two equations follow from the Lagrange density (1.97). Variation of the Monge potentials θ , α and β gives the current continuity equation (1.109) and equations satisfied by α and β

$$u^\mu \partial_\mu \alpha = 0 = u^\mu \partial_\mu \beta. \quad (1.111)$$

Variation of j^μ gives (1.100), whose curl reads

$$(\partial_\nu u_\mu - \partial_\mu u_\nu) f'(n) + (u_\mu \partial_\nu n - u_\nu \partial_\mu n) f''(n) = \partial_\mu a_\nu - \partial_\nu a_\mu = \partial_\mu \alpha \partial_\nu \beta - \partial_\nu \alpha \partial_\mu \beta. \quad (1.112)$$

Contracting this with u^μ makes the right-hand side vanish by virtue of (1.111) and the left-hand side coincides with (1.110).

It is especially intriguing to note that $\theta^{\mu\nu}$ is symmetric but $T^{\mu\nu}$ is not. To make the connection between the two we recall that $u^\mu = 1/\sqrt{1-\mathbf{v}^2/c^2}(1, \mathbf{v}/c)$, we observe that $n = \sqrt{\rho^2 c^2 - \mathbf{j}^2}$, set $\mathbf{j} = \mathbf{v}\rho$ and conclude that $n = \rho c \sqrt{1 - \mathbf{v}^2/c^2} \sim \rho c - (\rho \mathbf{v}^2/2c)$. Also $f(n)$ is chosen to be $cn + V(n/c)$, and thus $P = nf'(n) - f(n) = (n/c)V'(n/c) - V(n/c)$. It follows that

$$\begin{aligned} \theta^{00} &= \frac{nc - (\mathbf{v}^2 n/c^3)V'}{1 - \mathbf{v}^2/c^2} + V \approx \frac{\rho c^2 - \rho \mathbf{v}^2/2}{1 - \mathbf{v}^2/c^2} + V(\rho) \\ &\approx \rho c^2 + \frac{\rho \mathbf{v}^2}{2} + V(\rho) = \rho c^2 + T^{00}. \end{aligned} \quad (1.113)$$

Thus, apart from the relativistic 'rest energy' ρc^2 , θ^{00} passes to T^{00} . The relativistic energy flux is $c\theta^{j0}$ (because $\frac{\partial}{\partial x^\mu} \theta^{\mu 0} = \frac{1}{c} \dot{\theta}^{00} + \partial_j \theta^{j0}$).

$$\begin{aligned} c\theta^{j0} &= \frac{v^j}{1 - \mathbf{v}^2/c^2} \left(nc + \frac{n}{c} V' \right) \approx v^j \frac{\rho c^2 - \rho \mathbf{v}^2/2 + \rho V'(\rho)}{1 - \mathbf{v}^2/c^2} \\ &\approx j^j c^2 + \rho v^j (\mathbf{v}^2/2 + V'(\rho)) = j^j c + T^{j0}. \end{aligned} \quad (1.114)$$

Again, apart from the $O(c^2)$ current, associated with the $O(c^2)$ rest energy in θ^{00}, T^{j0} is obtained in the limit. The momentum density is θ^{0i}/c (because $\theta^{\mu\nu}$ has dimension of energy density):

$$\theta^{0i}/c = \frac{v^i/c^2}{1 - \mathbf{v}^2/c^2} \left(nc + \frac{n}{c} V' \right) \approx \rho v^i = \mathcal{P}^i. \quad (1.115)$$

Finally, the momentum flux is obtained directly from θ^{ij} :

$$\begin{aligned} \theta^{ij} &= \delta^{ij} \left(\frac{n}{c} V' - V \right) + \frac{v^i v^j}{c^2 - \mathbf{v}^2} \left(nc + \frac{n}{c} V' \right) \\ &\approx \delta^{ij} (\rho V'(\rho) - V(\rho)) + v^i v^j \rho = T^{ij}. \end{aligned} \quad (1.116)$$

2. Specific models ($d \neq 1$)

We now examine irrotational models, both relativistic and nonrelativistic, for which we shall specify an explicit force law and discuss further properties.

2.1. Two models, mostly in spatial dimensions $d \neq 1$

2.1.1. *Galileo-invariant nonrelativistic model: Chaplygin gas.* Recall that the nonrelativistic Lagrangian for irrotational motion reads

$$L^{\text{Galileo}} = \int dr \left(\theta \dot{\rho} - \rho \frac{(\nabla\theta)^2}{2} - V(\rho) \right), \quad (2.1)$$

where $\nabla\theta = \mathbf{v}$. The Hamiltonian density \mathcal{H} is composed of the last two terms beyond the canonical 1-form $\int dr \theta \dot{\rho}$,

$$H = \int dr \left(\rho \frac{(\nabla\theta)^2}{2} + V(\rho) \right) = \int dr \mathcal{H}. \quad (2.2)$$

Varying (2.1) with respect to ρ produces the Bernoulli equation (1.95). Various expressions for V appear in the literature. $V(\rho) \propto \rho^n$ is a popular choice, appropriate for the adiabatic equation of state. We shall be specifically interested in the Chaplygin gas [5]:

$$V(\rho) = \lambda/\rho, \quad \lambda > 0. \quad (2.3)$$

According to what we mentioned before, the Chaplygin gas has enthalpy $V' = -\lambda/\rho^2$, negative pressure $P = -2\lambda/\rho$, and the speed of sound $s = \sqrt{2\lambda}/\rho$ (hence $\lambda > 0$).

Chaplygin introduced his equation of state as a mathematical approximation to the physically relevant adiabatic expressions with $n > 0$. (Constants are arranged so that the Chaplygin formula is tangent at one point to the adiabatic profile.) Also it was realized that certain deformable solids can be described by the Chaplygin equation of state. These days negative pressure is recognized as a possible physical effect: exchange forces in atoms give rise to negative pressure; stripe states in the quantum Hall effect may be a consequence of negative pressure; the recently discovered cosmological constant may be exerting negative pressure on the cosmos, thereby accelerating expansion.

For any form of V , the model possesses the Galileo symmetry, discussed previously as appropriate to nonrelativistic dynamics. There are a total of $\frac{1}{2}(d+1)(d+2)$ Galileo generators in d space plus one time dimension. Together with the central term, N , we have a total of $\frac{1}{2}(d+1)(d+2) + 1$ generators.

A useful consequence of symmetry transformations is that they map solutions of the equations of motion into new solutions. Of course, ‘new’ solutions produced by Galileo

transformations are trivially related to the old ones: they are simply shifted, boosted or rotated.

But we shall now turn to the specific Chaplygin gas model, with $V(\rho) = \lambda/\rho$, which possesses additional and unexpected symmetries.

The Chaplygin gas action and consequent Bernoulli equation for the Chaplygin gas in $(d, 1)$ spacetime read

$$I_\lambda^{\text{Chaplygin}} = \int dt \int dr \left(\theta \dot{\rho} - \rho \frac{(\nabla\theta)^2}{2} - \frac{\lambda}{\rho} \right) \quad (2.4)$$

$$\dot{\theta} + \frac{(\nabla\theta)^2}{2} = \frac{\lambda}{\rho^2}. \quad (2.5)$$

This model possesses further spacetime symmetries beyond those of the Galileo group⁶. First, there is a one-parameter (ω , dimensionless) time rescaling transformation

$$t \rightarrow T = e^\omega t, \quad (2.6)$$

under which the fields transform as

$$\theta(t, \mathbf{r}) \rightarrow \theta_\omega(t, \mathbf{r}) = e^\omega \theta(T, \mathbf{r}), \quad (2.7)$$

$$\rho(t, \mathbf{r}) \rightarrow \rho_\omega(t, \mathbf{r}) = e^{-\omega} \rho(T, \mathbf{r}). \quad (2.8)$$

Second, in d spatial dimensions, there is a vectorial, d -parameter ($\boldsymbol{\omega}$, dimension inverse velocity) spacetime mixing transformation:

$$t \rightarrow T(t, \mathbf{r}) = t + \boldsymbol{\omega} \cdot \mathbf{r} + \frac{1}{2} \boldsymbol{\omega}^2 \theta(T, \mathbf{R}) \quad (2.9)$$

$$\mathbf{r} \rightarrow \mathbf{R}(t, \mathbf{r}) = \mathbf{r} + \boldsymbol{\omega} \theta(T, \mathbf{R}). \quad (2.10)$$

Note that the transformation law for the coordinates involves the θ field itself. Under this transformation, the fields transform according to

$$\theta(t, \mathbf{r}) \rightarrow \theta_\omega(t, \mathbf{r}) = \theta(T, \mathbf{R}), \quad (2.11)$$

$$\rho(t, \mathbf{r}) \rightarrow \rho_\omega(t, \mathbf{r}) = \rho(T, \mathbf{R}) \frac{1}{|J|}, \quad (2.12)$$

with J the Jacobian of the transformation linking $(T, \mathbf{R}) \rightarrow (t, \mathbf{r})$.

$$J = \det \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial T}{\partial r^j} \\ \frac{\partial R^i}{\partial t} & \frac{\partial R^i}{\partial r^j} \end{pmatrix} = \left(1 - \boldsymbol{\omega} \cdot \nabla \theta(T, \mathbf{R}) - \frac{\boldsymbol{\omega}^2}{2} \dot{\theta}(T, \mathbf{R}) \right)^{-1}. \quad (2.13)$$

(The time and space derivatives in the last element are with respect to t and \mathbf{r} .) One can tell the complete story for these transformations: the action (2.4) is invariant; Noether's theorem gives the conserved quantities, which for the time rescaling is

$$S = tH - \int dr \rho \theta \quad (\text{time rescaling}), \quad (2.14)$$

⁶ These symmetry transformations were identified by Bazeia and Jackiw [16], on the basis of constants of motion that were found previously; see [17].

while for the spacetime mixing one finds

$$\mathbf{G} = \int d\mathbf{r}(\mathbf{r}\mathcal{H} - \theta\mathcal{P}) \quad (\text{spacetime mixing}). \quad (2.15)$$

The time independence of S and \mathbf{G} can be verified with the help of the equations of motion (continuity and Bernoulli) [17]⁷. Poisson bracketing the fields θ and ρ with S and \mathbf{G} generates the appropriate infinitesimal transformation on the fields. Note that unlike the Galileo constants of motion, the new constants of motion cannot be locally expressed in terms of \mathbf{v} : their integrands depend locally on $\theta = \frac{\nabla}{\sqrt{2}} \cdot \mathbf{v}$.

So now the total number of generators is the sum of the previous $\frac{1}{2}(d+1)(d+2) + 1$ with $1 + d$ additional ones:

$$\frac{1}{2}(d+1)(d+2) + 1 + 1 + d = \frac{1}{2}(d+2)(d+3). \quad (2.16)$$

When one computes the Poisson brackets of all these with each other one finds the Poincaré Lie algebra in one higher spatial dimension, that is, in $(d+1, 1)$ -dimensional spacetime, where the Poincaré group possesses $\frac{1}{2}(d+2)(d+3)$ generators. Moreover, one verifies that (t, θ, \mathbf{r}) transform linearly as a $(d+2)$ Lorentz vector in light-cone components, with t being the ‘+’ component and θ the ‘-’ component [18].

Presently, we shall use these additional symmetries to generate new solutions from old ones, but, in contrast to the Galileo transformations, the new solutions will be non-trivially linked to the former ones. Note that the additional symmetry holds even in the free theory.

Before proceeding, let us observe that ρ may be eliminated by using the Bernoulli equation to express it in terms of θ . In this way, one is led to the following ρ -independent action for θ in the Chaplygin gas problem:

$$I_\lambda^{\text{Chaplygin}} = -2\sqrt{\lambda} \int dt \int d\mathbf{r} \sqrt{\dot{\theta} + \frac{(\nabla\theta)^2}{2}}. \quad (2.17)$$

Although this operation is possible only in the interacting case, the interaction strength disappears from the equations of motion:

$$\frac{\partial}{\partial t} \frac{1}{\sqrt{\dot{\theta} + \frac{(\nabla\theta)^2}{2}}} + \nabla \cdot \frac{\nabla\theta}{\sqrt{\dot{\theta} + \frac{(\nabla\theta)^2}{2}}} = 0. \quad (2.18)$$

λ merely serves as an overall factor in the action.

The action (2.17) looks unfamiliar; yet it is Galileo invariant. (The combination $\dot{\theta} + \frac{1}{2}(\nabla\theta)^2$ responds to Galileo transformations without a 1-cocycle; see (1.74)). Also (2.17) possesses the additional symmetries described above, with θ transforming according to the previously recorded equations.

Let us discuss some solutions. For example, the free theory is solved by

$$\theta(t, \mathbf{r}) = \frac{\mathbf{r}^2}{2t} \quad (2.19)$$

which corresponds to the velocity

$$\mathbf{v}(t, \mathbf{r}) = \frac{\mathbf{r}}{t}. \quad (2.20)$$

Galileo transformations generalize this in an obvious manner into a set of displaced, rotated and boosted solutions. Performing on the above formula for θ the new transformations of time rescaling and spacetime mixing (2.6), (2.7), (2.9)–(2.11), we find that the solution is invariant.

⁷ These authors consider only the planar, $d = 2$, case.

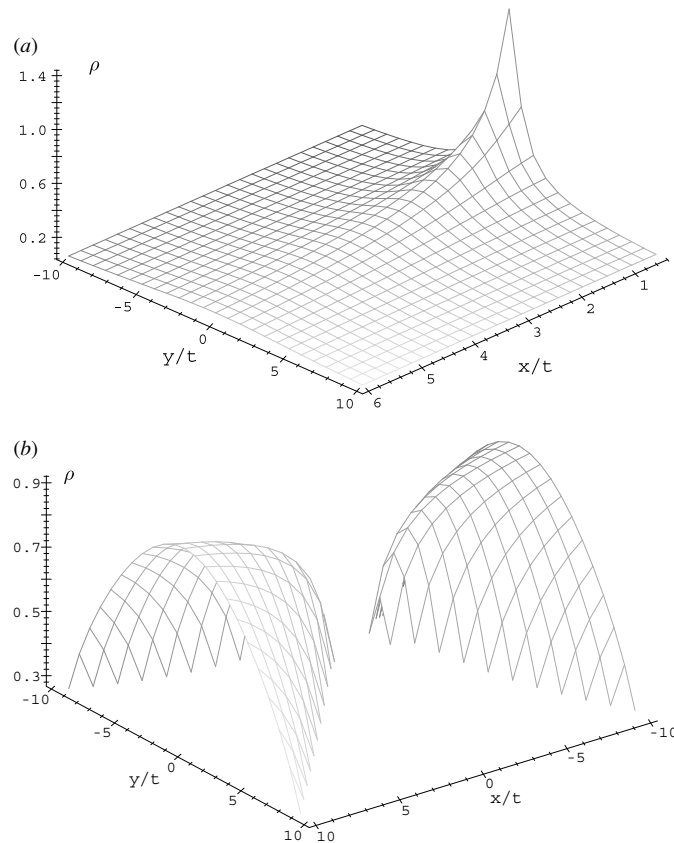


Figure 1. (a) The original density $\rho(t, r) \propto \frac{|t|}{r}$ (in two spatial dimensions, $r = (x, y)$). (b) The transformed density $\rho(t, r)$.

We can find a solution similar to (2.19) in the interacting case, for $d > 1$, which we henceforth assume. (The $d = 1$ case will be separately discussed in section 3.) One verifies that a solution is

$$\theta(t, \mathbf{r}) = -\frac{\mathbf{r}^2}{2(d-1)t} \quad \rho(t, \mathbf{r}) = \sqrt{\frac{2\lambda}{d}}(d-1)\frac{|t|}{r} = \sqrt{\frac{2\lambda}{d}}\frac{1}{v} \quad (2.21)$$

$$\mathbf{v}(t, \mathbf{r}) = -\frac{\mathbf{r}}{(d-1)t} \quad \mathbf{j}(t, \mathbf{r}) = -\epsilon(t)\sqrt{\frac{2\lambda}{d}}\hat{\mathbf{r}}. \quad (2.22)$$

Note that the speed of sound, $s = \sqrt{2\lambda/\rho} = \sqrt{d}v$, exceeds v . Again this solution can be translated, rotated and boosted. Moreover, the solution is time rescaling-invariant. However, the spacetime mixing transformation (2.9)–(2.13) produces a wholly different solution. This is best shown graphically, where the $d = 2$ case is exhibited (see figure 1).⁸

Another interesting solution, which is essentially one dimensional (lineal), even though it exists in arbitrary spatial dimension, is given by

⁸ Analytic details for implementing the spacetime mixing transformation on (2.21) are presented by Bazeia and Jackiw [16].

$$\begin{aligned} \theta(t, \mathbf{r}) &= \Theta(\hat{\mathbf{n}} \cdot \mathbf{r}) + \mathbf{u} \cdot \mathbf{r} - \frac{1}{2}t(\mathbf{u}^2 - (\hat{\mathbf{n}} \cdot \mathbf{u})^2). \\ \mathbf{v}(t, \mathbf{r}) &= \mathbf{u} + \hat{\mathbf{n}}\Theta'(\hat{\mathbf{n}} \cdot \mathbf{r}). \end{aligned} \tag{2.23}$$

Here $\hat{\mathbf{n}}$ is a spatial unit vector, \mathbf{u} is an arbitrary vector with dimension of velocity, while θ is an arbitrary function with static argument. The corresponding charge density is time independent:

$$\rho(t, \mathbf{r}) = \frac{\sqrt{2\lambda}}{|\hat{\mathbf{n}} \cdot \mathbf{u} + \Theta'(\hat{\mathbf{n}} \cdot \mathbf{r})|} = \frac{2\lambda}{|\hat{\mathbf{n}} \cdot \mathbf{v}|}. \tag{2.24}$$

The current is static and divergenceless:

$$\mathbf{j}(t, \mathbf{r}) = \sqrt{2\lambda} \left(\frac{\mathbf{v}}{|\hat{\mathbf{n}} \cdot \mathbf{v}|} \right). \tag{2.25}$$

The speed of sound $s = \sqrt{2\lambda}/\rho = |\hat{\mathbf{n}} \cdot \mathbf{v}|$ is smaller than v .

Finally, we record a planar static solution to (2.18), which depends on two orthogonal unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ [19]⁹

$$\begin{aligned} \theta(t, \mathbf{r}) &= \Theta(\hat{\mathbf{n}}_1 \cdot \mathbf{r}/\hat{\mathbf{n}}_2 \cdot \mathbf{r}) \\ \mathbf{v}(t, \mathbf{r}) &= \frac{(\hat{\mathbf{n}}_1(\hat{\mathbf{n}}_2 \cdot \mathbf{r}) - \hat{\mathbf{n}}_2(\hat{\mathbf{n}}_1 \cdot \mathbf{r}))}{(\hat{\mathbf{n}}_2 \mathbf{r})^2} \Theta'(\hat{\mathbf{n}}_1 \cdot \mathbf{r}/\hat{\mathbf{n}}_2 \cdot \mathbf{r}). \end{aligned} \tag{2.26}$$

This gives the density

$$\rho = \frac{\sqrt{2\lambda}}{v}. \tag{2.27}$$

Now the speed of sound coincides with v .

2.1.2. Lorentz-invariant relativistic model: Born-Infeld model. We now turn to a Lorentz-invariant generalization of our Galileo-invariant Chaplygin model in $(d, 1)$ -dimensional spacetime. We already know from (1.76)–(1.81) how to construct the free Lagrangian with a relativistic kinetic energy:

$$T(\mathbf{v}) = -c^2 \sqrt{1 - \mathbf{v}^2/c^2}. \tag{2.28}$$

Mass has been scaled to unity, and we retain the velocity of light c to keep track of the nonrelativistic $c \rightarrow \infty$ limit. Evidently, the momentum is

$$\mathbf{p} = \frac{\partial T(\mathbf{v})}{\partial \mathbf{v}} = \frac{\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}. \tag{2.29}$$

Thus the free relativistic Lagrangian, with current conservation enforced by the Lagrange multiplier θ , reads (compare (1.78), (1.79) with potentials α and β omitted)

$$\bar{L}_0^{\text{Lorentz}} = \int d\mathbf{r} \left(-c^2 \rho \sqrt{1 - \mathbf{v}^2/c^2} + \theta(\dot{\rho} + \nabla \cdot (\mathbf{v}\rho)) \right). \tag{2.30}$$

This may be presented in a Lorentz-covariant form in terms of a current 4-vector $j^\mu = (c\rho, \mathbf{v}\rho)$. $\bar{L}_0^{\text{Lorentz}}$ of equation (2.30) is thus equivalent to (compare (1.107)

$$\bar{L}_0^{\text{Lorentz}} = \int d\mathbf{r} \left(-j^\mu \partial_\mu \theta - c \sqrt{j^\mu j_\mu} \right). \tag{2.31}$$

Eliminating \mathbf{v} in (2.30), we find that \mathbf{p} is irrotational,

$$\mathbf{p} = \frac{\partial T}{\partial \mathbf{v}} = \frac{\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}} = \nabla \theta, \quad \mathbf{v} = \frac{\nabla \theta}{\sqrt{1 + (\nabla \theta)^2/c^2}}, \tag{2.32}$$

⁹ Ogawa also presents other solution, as does Bazeia [18].

and the free Lorentz-invariant Lagrangian reads

$$L_0^{\text{Lorentz}} = \int dr (\theta \dot{\rho} - \rho c^2 \sqrt{1 + (\nabla\theta)^2/c^2}). \quad (2.33)$$

To find L_0^{Galileo} ((2.1) with $V = 0$) as the nonrelativistic limit of L_0^{Lorentz} in (2.33), a nonrelativistic θ variable must be extracted from its relativistic counterpart. Calling the former θ_{NR} and the latter, which occurs in (2.33), θ_{R} , we define

$$\theta_{\text{R}} \equiv -c^2 t + \theta_{\text{NR}}. \quad (2.34)$$

It then follows that apart from a total time derivative

$$L_0^{\text{Lorentz}} \xrightarrow{c \rightarrow \infty} L_0^{\text{Galileo}}. \quad (2.35)$$

Next, one wants to include interactions. While there are many ways to build Lorentz-invariant interactions, we seek an expression that reduces to the Chaplygin gas in the nonrelativistic limit. Thus, we choose

$$L_a^{\text{Born-Infeld}} = \int dr (\theta \dot{\rho} - \sqrt{\rho^2 c^2 + a^2} \sqrt{c^2 + (\nabla\theta)^2}), \quad (2.36)$$

where a is the interaction strength [20]. (The reason for the nomenclature will emerge presently.) We see from (2.4) that, as $c \rightarrow \infty$,

$$L_a^{\text{Born-Infeld}} \xrightarrow{c \rightarrow \infty} L_{\lambda=a^2/2}^{\text{Chaplygin}}. \quad (2.37)$$

(Again θ_{NR} is extracted from θ_{R} as in (2.34) and a total time derivative is ignored.)

Although it perhaps is not obvious, (2.36) defines a Poincaré-invariant theory, and this will be explicitly demonstrated below. Therefore, $L_a^{\text{Born-Infeld}}$ possesses Poincaré symmetries in $(d, 1)$ spacetime, with a total of $\frac{1}{2}(d+1)(d+2) + 1$ generators, where the last ‘+ 1’ refers to the total number $N = \int dr \rho$.

When $a = 0$, the model is free and elementary. It was demonstrated previously (equations (1.84)) that the free equations of motion are precisely the same as in the nonrelativistic free model, so the complete solution (1.85)–(1.89) works here as well. For $a \neq 0$, in the presence of interactions, one can eliminate ρ as before, and one is left with a Lagrangian just for the θ field. It reads

$$L_a^{\text{Born-Infeld}} = -a \int dr \sqrt{c^2 - (\partial_\mu \theta)^2}. \quad (2.38)$$

This is a Born–Infeld-type theory for a scalar field θ ; its Poincaré invariance is manifest, and again, the elimination of ρ is only possible with non-vanishing a , which however disappears from the dynamics, serving merely to normalize the Lagrangian.

Although manifestly Lorentz covariant, the Lagrangian (2.38) is not of the form (1.97). To achieve that expression, we choose $f(n) = c\sqrt{a^2 + n^2}$, so that the pressure P becomes

$$P = -\frac{a^2 c}{\sqrt{a^2 + n^2}}. \quad (2.39)$$

This leads to the Chaplygin pressure as $c \rightarrow \infty$ with $\lambda = a^2/2$. Now the Lagrange density $\mathcal{L} = P$ coincides with (1.97) when j_μ is written as

$$j_\mu = \frac{a \partial_\mu \theta}{\sqrt{c^2 - (\partial_\mu \theta)^2}}, \quad n = a \sqrt{\frac{(\partial_\mu \theta)^2}{c^2 - (\partial_\mu \theta)^2}}. \quad (2.40)$$

The equations of motion that follow from (2.36) read

$$\dot{\rho} + \nabla \cdot \left(\nabla \theta \sqrt{\frac{\rho^2 c^2 + a^2}{c^2 + (\nabla \theta)^2}} \right) = 0, \tag{2.41}$$

$$\dot{\theta} + \rho c^2 \sqrt{\frac{c^2 + (\nabla \theta)^2}{\rho^2 c^2 + a^2}} = 0. \tag{2.42}$$

The density ρ can be evaluated in terms of θ from (2.42); then (2.41) becomes

$$\partial^\alpha \left(\frac{1}{\sqrt{c^2 - (\partial_\mu \theta)^2}} \partial_\alpha \theta \right) = 0, \tag{2.43}$$

which also follows from (2.38). After θ_{NR} is extracted from θ_{R} as in (2.34), we see that in the nonrelativistic limit $L_a^{\text{Born-Infeld}}$ (2.36) or (2.38) becomes $L_\lambda^{\text{Chaplygin}}$ of (2.4) or (2.17),

$$L_a^{\text{Born-Infeld}} \xrightarrow{c \rightarrow \infty} L_{\lambda=a^2/2}^{\text{Chaplygin}}, \tag{2.44}$$

and the equations of motion (2.41)–(2.43) reduce to (2.5) and (2.18).

In view of all the similarities to the nonrelativistic Chaplygin gas, it comes as no surprise that the relativistic Born–Infeld theory possesses additional symmetries. These additional symmetry transformations, which leave (2.36) or (2.38) invariant, involve a one-parameter (ω , dimensionless) reparametrization of time, and a d -parameter (ω , dimension velocity) vectorial reparametrization of space. Both transformations are field dependent¹⁰.

The time transformation is given by an implicit formula involving also the field θ ,

$$t \rightarrow T(t, \mathbf{r}) = \frac{t}{\cosh \omega} - \frac{\theta(T, \mathbf{r})}{c^2} \tanh \omega, \tag{2.45}$$

while the field transforms according to

$$\theta(t, \mathbf{r}) \rightarrow \theta_\omega(t, \mathbf{r}) = \frac{\theta(T, \mathbf{r})}{\cosh \omega} + c^2 t \tanh \omega. \tag{2.46}$$

(We record here only the transformation on θ ; how ρ transforms can be determined from the (relativistic) Bernoulli equation (2.42) which expresses ρ in terms of θ . Moreover, (2.46) is sufficient for discussing the invariance of (2.38).) The infinitesimal generator, which is time independent by virtue of the equations of motion, is [21]

$$\begin{aligned} S &= \int d\mathbf{r} (c^4 t \rho + \theta \sqrt{\rho^2 c^2 + a^2} \sqrt{c^2 + (\nabla \theta)^2}), \\ &= \int d\mathbf{r} (c^4 t \rho + \theta \mathcal{H}) \quad (\text{time reparametrization}). \end{aligned} \tag{2.47}$$

A second class of transformations involving a reparametrization of the spatial variables is implicitly defined by

$$\mathbf{r} \rightarrow \mathbf{R}(t, \mathbf{r}) = \mathbf{r} + \frac{\omega}{c^2} \theta(t, \mathbf{R}) + \hat{\omega} (\hat{\omega} \cdot \mathbf{r}) \left(\sqrt{1 + \frac{\omega^2}{c^2}} - 1 \right), \tag{2.48}$$

$$\theta(t, \mathbf{r}) \rightarrow \theta_\omega(t, \mathbf{r}) = \sqrt{1 + \frac{\omega^2}{c^2}} \theta(t, \mathbf{R}) + \omega \cdot \mathbf{r}. \tag{2.49}$$

¹⁰ The additional transformation rules were derived in [20], on the basis of constants of motion identified previously in the $d = 2$ case; see [21].

The time-independent generator of the infinitesimal transformation reads [21]

$$\begin{aligned}\mathbf{G} &= \int d\mathbf{r} (c^2 \mathbf{r} \rho + \theta \rho \nabla \theta), \\ &= \int d\mathbf{r} (c^2 \mathbf{r} \rho + \theta \mathcal{P}) \quad (\text{space reparametrization}).\end{aligned}\quad (2.50)$$

Of course the Born–Infeld action (2.36) or (2.38) is invariant against these transformations, whose infinitesimal form is generated by the constants.

With the addition of S and \mathbf{G} to the previous generators, the Poincaré algebra in $(d + 1, 1)$ dimension is reconstructed, and (t, \mathbf{r}, θ) transforms linearly as a $(d + 2)$ -dimensional Lorentz vector (in Cartesian components). Note that this symmetry also holds in the free, $a = 0$, theory.

It is easy to exhibit solutions of the relativistic equation (2.43), which reduce to solutions of the nonrelativistic, Chaplygin gas equation (2.18) (after $-c^2 t$ has been removed, as in (2.34)). For example,

$$\theta(t, \mathbf{r}) = -c \sqrt{c^2 t^2 + \frac{\mathbf{r}^2}{d-1}} \quad (2.51)$$

solves (2.43) and reduces to (2.21). The relativistic analogue of the lineal solution (2.23) is

$$\theta(t, \mathbf{r}) = \Theta(\hat{\mathbf{n}} \cdot \mathbf{r}) + \mathbf{u} \cdot \mathbf{r} - ct \sqrt{c^2 + \mathbf{u}^2 - (\hat{\mathbf{n}} \cdot \mathbf{u})^2}. \quad (2.52)$$

(Note that the above profiles continue to solve (2.43) even when the sign of the square root is reversed, but then they no longer possess a nonrelativistic limit.)

Additionally there exists an essentially relativistic solution, describing massless propagation in one direction: according to (2.43), θ can satisfy the wave equation $\square \theta = 0$, provided $(\partial_\mu \theta)^2 = \text{constant}$, as for example with plane waves,

$$\theta(t, \mathbf{r}) = f(\hat{\mathbf{n}} \cdot \mathbf{r} \pm ct), \quad (2.53)$$

where $(\partial_\mu \theta)^2$ vanishes. Then ρ reads, from (2.42),

$$\rho = \mp \frac{a}{c^2} f'. \quad (2.54)$$

2.2. Common ancestry: the Nambu–Goto action

The ‘hidden’ symmetries and the associated transformation laws for the Chaplygin and Born–Infeld models may be given a coherent setting by considering the Nambu–Goto action for a p -brane in $(p + 1)$ spatial dimensions, moving on $(p + 1, 1)$ -dimensional spacetime. In our context, a p -brane is simply a p -dimensional extended object: a 1-brane is a string, a 2-brane is a membrane and so on. A p -brane in $(p + 1)$ space divides that space in two.

The Nambu–Goto action reads

$$I_{\text{NG}} = \int d\phi^0 d\phi \mathcal{L}_{\text{NG}} = \int d\phi^0 d\phi^1 \cdots d\phi^p \sqrt{G}, \quad (2.55)$$

$$G = (-1)^p \det \frac{\partial X^\mu}{\partial \phi^\alpha} \frac{\partial X_\mu}{\partial \phi^\beta}. \quad (2.56)$$

Here X^μ is a $(p + 1, 1)$ ‘target spacetime’ (p -brane) variable, with μ extending over the range $\mu = 0, 1, \dots, p, p + 1$. The ϕ^α are ‘world-volume’ variables describing the extended object with α ranging $\alpha = 0, 1, \dots, p$; ϕ^r , $r = 1, \dots, p$, parametrizes the p -dimensional p -brane that evolves in ϕ^0 .

The Nambu–Goto action is parametrization invariant, and we shall show that two different choices of parametrization (‘light-cone’ and ‘Cartesian’) lead to the Chaplygin gas and Born–Infeld actions, respectively. For both parametrizations we choose (X^1, \dots, X^p) to coincide with (ϕ^1, \dots, ϕ^p) , renaming them as \mathbf{r} (a p -dimensional vector). This is usually called the ‘static parametrization’. (The ability to carry out this parametrization globally presupposes that the extended object is topologically trivial; in the contrary situation, singularities will appear, which are spurious in the sense that they disappear in different parametrizations, and parametrization-invariant quantities are singularity free.)

2.2.1. Light-cone parametrization. For the light-cone parametrization we define X^\pm as $\frac{1}{\sqrt{2}}(X^0 \pm X^{p+1})$. X^+ is renamed t and identified with $\sqrt{2\lambda}\phi^0$. This completes the fixing of the parametrization and the remaining variable is X^- , which is a function of ϕ^0 and ϕ , or after redefinitions, of t and \mathbf{r} . X^- is renamed as $\theta(t, \mathbf{r})$ and then the Nambu–Goto action in this parametrization coincides with the Chaplygin gas action $I_\lambda^{\text{Chaplygin}}$ in (2.17) [22]¹¹.

2.2.2. Cartesian parametrization. For the second, Cartesian parametrization X^0 is renamed ct and identified with $c\phi^0$. The remaining target space variable X^{p+1} , a function of ϕ^0 and ϕ , equivalently of t and \mathbf{r} , is renamed $\theta(t, \mathbf{r})/c$. Then the Nambu–Goto action reduces to the Born–Infeld action $\int dt L_a^{\text{Born–Infeld}}$, (2.38) [22].

2.2.3. Hodographic transformation. There is another derivation of the Chaplygin gas from the Nambu–Goto action that makes use of a hodographic transformation, in which independent and dependent variables are interchanged. Although the derivation is more involved than the light-cone/static parametrization used in section 2.2.1 above, the hodographic approach is instructive in that it gives a natural definition for the density ρ , which in the above static parametrization approach is determined from θ by the Bernoulli equation (2.5).

We again use light-cone combinations: $\frac{1}{\sqrt{2}}(X^0 + X^{p+1})$ is called τ and is identified with ϕ^0 , while $\frac{1}{\sqrt{2}}(X^0 - X^{p+1})$ is renamed θ . At this stage the dependent, target-space variables are θ and the transverse coordinates \mathbf{X} : X^i ($i = 1, \dots, p$), and all are functions of the world-volume parameters $\phi^0 = \tau$ and ϕ : ϕ^r ($r = 1, \dots, p$); ∂_τ indicates differentiation with respect to $\tau = \phi^0$, while ∂_r denotes derivatives with respect to ϕ^r . The induced metric $G_{\alpha\beta} = \frac{\partial X^\mu}{\partial \phi^\alpha} \frac{\partial X_\mu}{\partial \phi^\beta}$ takes the form

$$G_{\alpha\beta} = \begin{pmatrix} G_{00} & G_{0s} \\ G_{r0} & -g_{rs} \end{pmatrix} = \begin{pmatrix} 2\partial_\tau\theta - (\partial_\tau\mathbf{X})^2 & \partial_s\theta - \partial_\tau\mathbf{X} \cdot \partial_s\mathbf{X} \\ \partial_r\theta - \partial_r\mathbf{X} \cdot \partial_\tau\mathbf{X} & -\partial_r\mathbf{X} \cdot \partial_s\mathbf{X} \end{pmatrix}. \quad (2.57)$$

The Nambu–Goto Lagrangian now leads to the canonical momenta

$$\frac{\partial \mathcal{L}_{\text{NG}}}{\partial \partial_\tau \mathbf{X}} = \mathbf{p}, \quad (2.58)$$

$$\frac{\partial \mathcal{L}_{\text{NG}}}{\partial \partial_\tau \theta} = \Pi, \quad (2.59)$$

and can be presented in first-order form as

$$\mathcal{L}_{\text{NG}} = \mathbf{p} \cdot \partial_\tau \mathbf{X} + \Pi \partial_\tau \theta + \frac{1}{2\Pi} (\mathbf{p}^2 + g) + u^r (\mathbf{p} \cdot \partial_r \mathbf{X} + \Pi \partial_r \theta), \quad (2.60)$$

¹¹ That the theory of a membrane ($(d = 2)$ -brane) in three spatial dimensions is equivalent to planar fluid mechanics was known to Goldstone (unpublished) and worked out by his student Hoppe (sometimes in collaboration with Bordemann).

where $g = \det g_{rs}$ and

$$u_r \equiv \partial_\tau \mathbf{X} \cdot \partial_r \mathbf{X} - \partial_r \theta \quad (2.61)$$

acts as a Lagrange multiplier. Evidently, the equations of motion are

$$\partial_\tau \mathbf{X} = -\frac{1}{\Pi} \mathbf{p} - u^r \partial_r \mathbf{X}, \quad (2.62a)$$

$$\partial_\tau \theta = \frac{1}{2\Pi^2} (\mathbf{p}^2 + g) - u^r \partial_r \theta, \quad (2.62b)$$

$$\partial_\tau \mathbf{p} = -\partial_r \left(\frac{1}{\Pi} g g^{rs} \partial_s \mathbf{X} \right) - \partial_r (u^r \mathbf{p}), \quad (2.62c)$$

$$\partial_\tau \Pi = -\partial_r (u^r \Pi). \quad (2.62d)$$

Also there is the constraint

$$\mathbf{p} \cdot \partial_r \mathbf{X} + \Pi \partial_r \theta = 0. \quad (2.63)$$

(That u^r is still given by (2.61) is a consequence of (2.62a) and (2.63).) Here g^{rs} is inverse of g_{rs} , and the two metrics are used to move the (r, s) indices. The theory still possesses an invariance against redefining the spatial parameters with a τ -dependent function of the parameters; infinitesimally: $\delta\phi^r = -f^r(\tau, \phi)$, $\delta\theta = f^r \partial_r \theta$, $\delta X^i = f^r \partial_r X^i$. This freedom may be used to set u^r to zero and Π to -1 .

Next the hodographic transformation is performed: rather than viewing the dependent variables \mathbf{p} , θ and \mathbf{X} as functions of τ and ϕ , $\mathbf{X}(\tau, \phi)$ is inverted so that ϕ becomes a function of τ and \mathbf{X} (renamed t and \mathbf{r} , respectively), and \mathbf{p} and θ also become functions of t and \mathbf{r} . It then follows from the chain rule that the constraint (2.63) (at $\Pi = -1$) becomes

$$0 = \frac{\partial X^i}{\partial \phi^r} \left(p^i - \frac{\partial}{\partial X^i} \theta \right), \quad (2.64)$$

and is solved by

$$\mathbf{p} = \nabla \theta. \quad (2.65)$$

Moreover, according to the chain rule and the implicit function theorem, the partial derivative with respect to τ at fixed ϕ (this derivative is present in (2.60)) is related to the partial derivative with respect to τ at fixed $\mathbf{X} = \mathbf{r}$ by

$$\partial_\tau = \partial_t + \nabla \theta \cdot \nabla, \quad (2.66)$$

where we have used the new name ‘ t ’ on the right-hand side. Thus the Nambu–Goto Lagrangian—the ϕ integral of the Lagrange density (2.60) (at $u^r = 0$, $\Pi = -1$)—reads

$$L_{\text{NG}} = \int d\phi \left(\mathbf{p} \cdot \nabla \theta - \dot{\theta} - \nabla \theta \cdot \nabla \theta - \frac{1}{2} (\mathbf{p}^2 + g) \right). \quad (2.67a)$$

But use of (2.65) and of the Jacobian relation $d\phi = d\mathbf{r} \det \frac{\partial \phi^s}{\partial X^i} = \frac{d\mathbf{r}}{\sqrt{g}}$ shows that

$$L_{\text{NG}} = \int d\mathbf{r} \left(-\frac{1}{\sqrt{g}} \dot{\theta} - \frac{1}{2\sqrt{g}} (\nabla \theta)^2 - \frac{1}{2} \sqrt{g} \right). \quad (2.67b)$$

With the definition,

$$\sqrt{g} = \sqrt{2\lambda}/\rho, \quad (2.67c)$$

L_{NG} becomes, apart from a total time derivative,

$$L_{\text{NG}} = \frac{1}{\sqrt{2\lambda}} \int dr \left(\theta \dot{\rho} - \rho \frac{(\nabla\theta)^2}{2} - \frac{\lambda}{\rho} \right). \quad (2.67d)$$

Up to an overall factor, this is just the Chaplygin gas Lagrangian in (2.4).

The present derivation has the advantage of relating the density ρ to the Jacobian of the $\mathbf{X} \rightarrow \phi$ transformation: $\rho = \sqrt{2\lambda} \det \frac{\partial \phi^s}{\partial X^i}$. (This in turn shows that the hodographic transformation is just exactly the passage from Lagrangian to Eulerian fluid variables.)

2.3. Interrelations

The relation to the Nambu–Goto action explains the origin of the hidden $(p + 1, 1)$ Poincaré group in our two non-linear models on $(p, 1)$ spacetime: the $(p + 1, 1)$ Poincaré invariance is what remains of the reparametrization invariance of the Nambu–Goto action after choosing either the light-cone or Cartesian parametrizations. (In this context, recall that the light-cone subgroup of the Poincaré group is isomorphic to the Galileo group in one lower dimension [23].) Also the non-linear, field-dependent form of the transformation laws leading to the additional symmetries is understood: it arises from the identification of some of the dependent variables (X^μ) with the independent variables (ϕ^α).

The complete integrability of the $d = 1$ Chaplygin gas and Born–Infeld model is a consequence of the fact that both descend from a string in 2-space, the associated Nambu–Goto theory being completely integrable. We shall discuss this in section 3.

We observe that in addition to the nonrelativistic descent from the Born–Infeld theory to the Chaplygin gas, there exists a mapping of one system on another, and between solutions of one system and the other, because both have the same p -brane ancestor. The mapping is achieved by passing from the light-cone parametrization to the Cartesian, or vice versa. Specifically this is accomplished as follows.

2.3.1. Chaplygin gas \rightarrow Born–Infeld. Given $\theta_{\text{NR}}(t, \mathbf{r})$, a nonrelativistic solution, determine $T(t, \mathbf{r})$ from the equation

$$T + \frac{1}{c^2} \theta_{\text{NR}}(T, \mathbf{r}) = \sqrt{2}t. \quad (2.68)$$

Then the relativistic solution is

$$\theta_{\text{R}}(t, \mathbf{r}) = \frac{1}{\sqrt{2}} c^2 T - \frac{1}{\sqrt{2}} \theta_{\text{NR}}(T, \mathbf{r}) = c^2(\sqrt{2}T - t). \quad (2.69)$$

2.3.2. Born–Infeld \rightarrow Chaplygin gas. Given $\theta_{\text{R}}(t, \mathbf{r})$, a relativistic solution, find $T(t, \mathbf{r})$ from

$$T + \frac{1}{c^2} \theta_{\text{R}}(T, \mathbf{r}) = \sqrt{2}t. \quad (2.70)$$

Then the nonrelativistic solution is

$$\theta_{\text{NR}}(t, \mathbf{r}) = \frac{1}{\sqrt{2}} c^2 T - \frac{1}{\sqrt{2}} \theta_{\text{R}}(T, \mathbf{r}) = c^2(\sqrt{2}T - t). \quad (2.71)$$

The relation between the different models is depicted in figure 2 below.

A final comment. Recall that the elimination of ρ , both in the nonrelativistic (Chaplygin) and relativistic (Born–Infeld) models is possible only in the presence of interactions. Nevertheless, the θ -dependent (ρ -independent) resultant Lagrangians contain the interaction

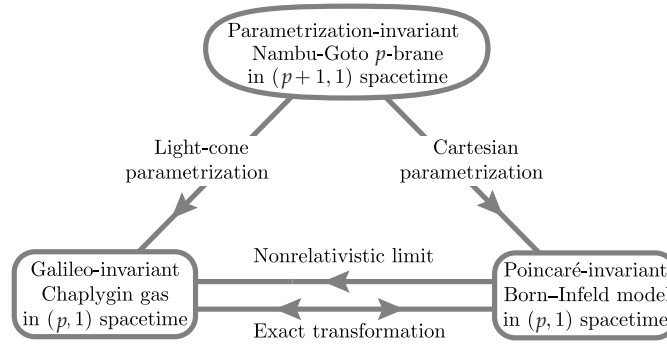


Figure 2. Dualities and other relations between non-linear equations.

strengths only as overall factors; see (2.17) and (2.38). It is these θ -valued Lagrangians that correspond to the Nambu–Goto action in various parametrizations. Let us further recall that the Nambu–Goto action also carries an overall multiplicative factor: the p -brane ‘tension’, which has been suppressed in (2.55). Correspondingly, for a ‘tensionless’ p -brane, the Nambu–Goto expression vanishes, and cannot generate dynamics. This suggests that an action for ‘tensionless’ p -branes could be the non-interacting fluid mechanical expressions (2.4), (2.36), with vanishing coupling strengths λ and a , respectively. Furthermore, we recall that the non-interacting models retain the higher, dynamical symmetries, appropriate to a p -brane in one higher dimension.

3. Specific models ($d = 1$)

In this section, we shall discuss nonrelativistic/relativistic models in one spatial dimension. Complete integrability has been established for both the Chaplygin gas [2, 24] and the Born–Infeld theory [25]. We can now understand this to be a consequence of the complete integrability of the Nambu–Goto 1-brane (string) moving on 2-space (plane), which is the antecedent of both models. (Therefore, it suffices to discuss only the Chaplygin gas, since solutions of the Born–Infeld model can then be obtained by the mapping (2.68)–(2.69).)

As remarked previously, in one dimension there is no vorticity, and the nonrelativistic velocity v can be presented as a derivative with respect to the single spatial variable of a potential θ . Similarly, the relativistic momentum $p = v/\sqrt{1 - v^2/c^2}$ is a derivative of a potential θ . In both cases the potential is canonically conjugate to the density ρ governed by the canonical 1-form $\int dx \theta \dot{\rho}$. Moreover, it is evident that at the expense of a spatial non-locality, one may replace θ by its antiderivative, which is p both nonrelativistically and relativistically (nonrelativistically $p = v$), so that in both cases the Lagrangian reads

$$L = -\frac{1}{2} \int dx dy \rho(x) \varepsilon(x - y) \dot{p}(y) - H. \tag{3.1}$$

For the Chaplygin gas and the Born–Infeld models, H is given respectively by

$$H^{\text{Chaplygin}} = \int dx \left(\frac{1}{2} \rho p^2 + \frac{\lambda}{\rho} \right) \tag{3.2}$$

$$H^{\text{Born-Infeld}} = \int dx (\sqrt{\rho^2 c^2 + a^2} \sqrt{c^2 + p^2}). \tag{3.3}$$

The equations of motion are, respectively,

Chaplygin gas

$$\dot{\rho} + \frac{\partial}{\partial x}(p\rho) = 0, \tag{3.4}$$

$$\dot{p} + \frac{\partial}{\partial x}\left(\frac{p^2}{2} - \frac{\lambda}{\rho^2}\right) = 0, \tag{3.5}$$

or

$$\frac{\partial}{\partial t} \frac{1}{\sqrt{\dot{\theta} + \frac{v^2}{2}}} + \frac{\partial}{\partial x} \frac{p}{\sqrt{\dot{\theta} + \frac{v^2}{2}}} = 0. \tag{3.6}$$

Born–Infeld model

$$\dot{\rho} + \frac{\partial}{\partial x}\left(p\sqrt{\frac{\rho^2 c^2 + a^2}{c^2 + p^2}}\right) = 0, \tag{3.7}$$

$$\dot{p} + \frac{\partial}{\partial x}\left(\rho c^2 \sqrt{\frac{c^2 + p^2}{\rho^2 c^2 + a^2}}\right) = 0, \tag{3.8}$$

or

$$\frac{\partial}{c^2 \partial t} \left(\frac{\dot{\theta}}{\sqrt{c^2 - \frac{1}{c^2} \dot{\theta}^2 + p^2}} \right) - \frac{\partial}{\partial x} \left(\frac{p}{\sqrt{c^2 - \frac{1}{c^2} \dot{\theta}^2 + p^2}} \right) = 0. \tag{3.9}$$

In the above, equations (3.6) and (3.9) result by determining ρ in terms of θ ($p = \theta'$; prime indicates differentiation with respect to spatial argument) from (3.5) and (3.8), and using that expression for ρ in (3.4) and (3.7).

3.1. Chaplygin gas on a line

3.1.1. *Specific solutions.* Classes of solutions for a Chaplygin gas in one dimension can be given in closed form. For example, to obtain general, time-rescaling-invariant solutions, we make the ansatz that $\theta \propto 1/t$. Then (2.18) or (3.6) leads to a second-order non-linear differential equation for the x -dependence of θ . Therefore solutions involve two arbitrary constants, one of which fixes the origin of x (we suppress it); the other we call k , and take it to be real. The solutions then read

$$\theta(t, x) = -\frac{1}{2k^2 t} \cosh^2 kx. \tag{3.10}$$

(Other solutions can be obtained by relaxing the reality condition on k and/or shifting the argument kx by a complex number. In this way one finds that θ can also be $\frac{1}{2k^2 t} \sinh^2 kx$, $\frac{1}{2k^2 t} \sin^2 kx$, $\frac{1}{2k^2 t} \cos^2 kx$; but these lead to singular or unphysical forms for ρ .) The density corresponding to (3.10) is found from (2.5) or (3.5) to be

$$\rho(t, x) = \sqrt{2\lambda} \frac{k|t|}{\cosh^2 kx}. \tag{3.11}$$

The velocity/momentum $v = p = \theta'$ is

$$v(t, x) = p(t, x) = -\frac{1}{kt} \sinh kx \cosh kx, \tag{3.12}$$

while the speed of sound

$$s(t, x) = \frac{\cosh^2 kx}{k|t|}, \quad (3.13)$$

is always greater than $|v|$. Finally, the current $j = \rho \frac{\partial \theta}{\partial x}$ exhibits a kink profile,

$$j(t, x) = -\varepsilon(t) \sqrt{2\lambda} \tanh kx, \quad (3.14)$$

which is suggestive of complete integrability.

Another particular solution is the Galileo boost of the static profiles (2.24), (2.25):

$$p(t, x) = p(x - ut), \quad (3.15)$$

$$\rho(t, x) = \frac{\sqrt{2\lambda}}{|p - u|}. \quad (3.16)$$

Here u is the boosting velocity and $p(x - ut)$ is an arbitrary function of its argument (provided $p \neq u$). Clearly this is a constant profile solution, in linear motion with velocity u .¹²

Further evidence for complete integrability is found by identifying an infinite number of constants of motion. One verifies that the following quantities

$$I_n^\pm = \int dx \rho \left(p \pm \frac{\sqrt{2\lambda}}{\rho} \right)^n, \quad n = 0, \pm 1, \dots \quad (3.17)$$

are conserved.

The combinations $p \pm \frac{\sqrt{2\lambda}}{\rho}$ are just the velocity (\pm) the speed of sound, and they are known as Riemann coordinates:

$$R_\pm = p \pm \frac{\sqrt{2\lambda}}{\rho}. \quad (3.18)$$

The equations of motion for this system (continuity (3.4) and Euler (3.5)) can be succinctly presented in terms of R_\pm :

$$\dot{R}_\pm = -R_\mp R'_\pm. \quad (3.19)$$

3.1.2. General solution for the Chaplygin gas on a line. The general solution to the Chaplygin gas can be found by linearizing the governing equations (continuity and Euler) with the help of a Legendre transform, which also effects a hodographic transformation that exchanges the independent variables (t, x) with the dependent ones (ρ, θ) ; actually instead of ρ we use the speed of sound $s = \sqrt{2\lambda}/\rho$ and instead of θ we use the momentum $p = \theta'$.

Define

$$\psi(p, s) = \theta(t, x) - t\dot{\theta}(t, x) - x\theta'(t, x). \quad (3.20)$$

From the Bernoulli equation we know that

$$\dot{\theta} = -\frac{1}{2}p^2 + \frac{1}{2}s^2. \quad (3.21)$$

Thus

$$\psi(p, s) = \theta(t, x) + \frac{t}{2}(p^2 - s^2) - xp, \quad (3.22)$$

¹² Other solutions, enjoying group invariance or partial invariance, have been constructed by Grundland *et al* [26].

and the usual Legendre transform rules govern the derivatives:

$$\frac{\partial \psi}{\partial p} = tp - x \tag{3.23a}$$

$$\frac{\partial \psi}{\partial s} = -ts. \tag{3.23b}$$

It remains to incorporate the continuity equation (3.4) whose content must be recast by the hodographic transformation. This is achieved by rewriting equation (3.4) in terms of $s = \sqrt{2\lambda/\rho}$,

$$\frac{\partial s}{\partial t} + p \frac{\partial s}{\partial x} - s \frac{\partial p}{\partial x} = 0. \tag{3.24}$$

Next (3.24) is presented as a relation between Jacobians

$$\frac{\partial(s, x)}{\partial(t, x)} + p \frac{\partial(t, s)}{\partial(t, x)} - s \frac{\partial(t, p)}{\partial(t, x)} = 0, \tag{3.25a}$$

which is true because here $\partial x/\partial t = \partial t/\partial x = 0$. Equation (3.25a) implies, after multiplication by $\partial(t, x)/\partial(s, p)$

$$\begin{aligned} 0 &= \frac{\partial(s, x)}{\partial(s, p)} + p \frac{\partial(t, s)}{\partial(s, p)} - s \frac{\partial(t, p)}{\partial(s, p)} \\ &= \frac{\partial x}{\partial p} - p \frac{\partial t}{\partial p} - s \frac{\partial t}{\partial s}. \end{aligned} \tag{3.25b}$$

The second equality holds because now we take $\partial s/\partial p = \partial p/\partial s = 0$. Finally, from (3.22), (3.23) it follows that (3.25b) is equivalent to

$$\frac{\partial^2 \psi}{\partial p^2} - \frac{\partial^2 \psi}{\partial s^2} + \frac{2}{s} \frac{\partial \psi}{\partial s} = 0. \tag{3.25c}$$

This linear equation is solved by two arbitrary functions of $p \pm s$ ($p \pm s$ being just the Riemann coordinates):

$$\psi(p, s) = F(p + s) - sF'(p + s) + G(p - s) + sG'(p - s). \tag{3.26}$$

In summary, to solve the Chaplygin gas equations, we choose two functions F and G , construct ψ as in (3.26), and regain $s (= \sqrt{2\lambda/\rho})$, $p (= \theta')$, and θ from (3.22), (3.23). In particular, the solution (3.10), (3.11) corresponds to

$$F(z) = G(-z) = \pm \frac{z}{2k} \ln z, \tag{3.27}$$

where the sign is correlated with the sign of t .

3.2. Born–Infeld model on a line

Since the Born–Infeld system is related to the Chaplygin gas by the transformation described in section 2.3, there is no need to discuss separately Born–Infeld solutions. Nevertheless, the formulation in terms of Riemann coordinates is especially succinct and gives another view on the Chaplygin/Born–Infeld relation.

The Riemann coordinates R_{\pm} for the Born–Infeld model are constructed by first defining

$$\frac{1}{c} \theta' = p/c = \tan \phi_p, \quad a/\rho c = \tan \phi_\rho, \tag{3.28}$$

and

$$R_{\pm} = \phi_p \pm \phi_\rho. \tag{3.29}$$

The one-dimensional version of the equations of motion (2.41), (2.42), that is, (3.7), (3.8) can be presented as

$$\dot{R}_{\pm} = -c(\sin R_{\mp})R'_{\pm}. \quad (3.30)$$

The relation to the Riemann description of the Chaplygin gas can now be seen in two ways: a nonrelativistic limit and an exact transformation. For the former, we note that at large c , $\phi_p \approx p/c$, $\phi_\rho \approx a/\rho c$ so that

$$R_{\pm}^{\text{Born-Infeld}} \approx \frac{1}{c} \left(p \pm \frac{a}{\rho} \right) = \frac{1}{c} R_{\pm}^{\text{Chaplygin}} \Big|_{\lambda=a^2/2}. \quad (3.31)$$

Moreover, equation (3.30) becomes, in view of (3.31),

$$\frac{1}{c} \dot{R}_{\pm}^{\text{Chaplygin}} = -R_{\mp}^{\text{Chaplygin}} \frac{1}{c} \frac{\partial}{\partial x} R_{\pm}^{\text{Chaplygin}}, \quad (3.32)$$

so that (3.19) is regained. On the other hand, for the exact transformation we define new Riemann coordinates in the relativistic Born-Infeld case by

$$\mathcal{R}_{\pm} = c \sin R_{\pm}. \quad (3.33)$$

Evidently (3.30) implies that \mathcal{R}_{\pm} satisfies the nonrelativistic equations (3.19) and (3.32) when R_{\pm} solves the relativistic equation (3.30). Expressing \mathcal{R}_{\pm} and R_{\pm} in terms of the corresponding nonrelativistic and relativistic variables produces a mapping between the two sets. Calling p_{NR} , ρ_{NR} and p_{R} , ρ_{R} the momentum and density of the nonrelativistic and of the relativistic theory, respectively, the mapping implied by (3.33) is

$$p_{\text{NR}} = \frac{c^2 \rho_{\text{R}} p_{\text{R}}}{\sqrt{(p_{\text{R}}^2 + c^2)(\rho_{\text{R}}^2 c^2 + a^2)}}, \quad \rho_{\text{NR}} = \frac{1}{c^2} \sqrt{(p_{\text{R}}^2 + c^2)(\rho_{\text{R}}^2 c^2 + a^2)}. \quad (3.34)$$

As can be checked, this maps the Chaplygin equations into the Born-Infeld equations. But the mapping is not canonical.

We record the infinite number of constants of motion, which put into evidence the (by now obvious) complete integrability of the Born-Infeld equations on a line. The following quantities are time independent:

$$I_n^{\pm} = ac^{n-1} \int dx \frac{(\phi_p \pm \phi_\rho)^n}{\sin \phi_\rho \cos \phi_p}, \quad n = 0, \pm 1, \dots \quad (3.35)$$

The nonrelativistic limit takes the above into (3.17), while expressing I_n^{\pm} in terms of \mathcal{R}_{\pm} according to (3.33) shows that the integrals in (3.35) gives rise to a series of the integrals in (3.17).

In the relativistic model ρ need not be constrained to be positive (negative ρ could be interpreted as antiparticle density). The transformation $p \rightarrow -p$, $\rho \rightarrow -\rho$ is a symmetry and can be interpreted as charge conjugation.

Further, p and ρ appear in an equivalent way. As a result, this theory enjoys a duality transformation:

$$\rho \rightarrow \pm \frac{a}{c^2} p \quad p \rightarrow \pm \frac{c^2}{a} \rho. \quad (3.36)$$

Under the above, both the canonical structure and the Hamiltonian remain invariant. Solutions are mapped in general to new solutions. Note that the nonrelativistic limit is mapped to the ultra-relativistic one under the above duality. Self-dual solutions, with $\rho = \pm \frac{a}{c^2} p$, satisfy

$$\dot{\rho} = \mp c \frac{\partial}{\partial x} \rho', \quad (3.37)$$

and are, therefore, the chiral relativistic solutions that were presented at the end of section 2.1.2. In the self-dual case, when p is eliminated from the canonical 1-form and from the Hamiltonian with the help of (3.36), one arrives at an action for ρ , which coincides (apart from irrelevant constants) with the self-dual action, constructed some time ago [28]:

$$\left. \left\{ \frac{1}{2} \int dt dx dy \dot{\rho}(x) \epsilon(x-y) p(y) - \int dt dx \sqrt{\rho^2 c^2 + a^2} \sqrt{c^2 + p^2} dt \right\} \right|_{p=\frac{c^2}{a}\rho} = \frac{2c^2}{a} \left\{ \frac{1}{4} \int dt dx dy \dot{\rho}(x) \epsilon(x-y) \rho(y) - \frac{c}{2} \int dt dx \left(\rho^2(x) + \frac{a^2}{c^2} \right) \right\}. \quad (3.38)$$

3.3. General solution of the Nambu–Goto theory for a ($p = 1$)-brane (string) in two spatial dimensions (on a plane)

The complete integrability of the one-dimensional Chaplygin gas and Born–Infeld theory, as well as the relationships between the two, derives from the fact that the different models descend by fixing in different ways the parametrization invariance of the Nambu–Goto theory for a string on a plane. At the same time, the equations governing the planar motion of a string can be solved completely. Therefore it is instructive to see how the string solution produces the Chaplygin solution [18].

We follow the development in section 2.2.3. The Nambu–Goto action reads

$$I_{\text{NG}} = \int d\phi^0 L_{\text{NG}}, \quad (3.39a)$$

$$L_{\text{NG}} = \int d\phi^1 \mathcal{L}_{\text{NG}}, \quad (3.39b)$$

$$\mathcal{L}_{\text{NG}} = \left[-\det \frac{\partial X^\mu}{\partial \phi^\alpha} \frac{\partial X_\mu}{\partial \phi^\beta} \right]^{1/2}. \quad (3.39c)$$

Here $X^\mu, \mu = 0, 1, 2$, are string variables and (ϕ^0, ϕ^1) are its parameters. As in section 2.2.3, we define light-cone combinations $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^2)$, rename X^- as θ , and choose the parametrization $X^+ = \phi^0 \equiv \tau$. After suppressing the superscripts on ϕ^1 and X^1 , we construct the Nambu–Goto Lagrange density as

$$\mathcal{L}_{\text{NG}} = \det^{1/2} \begin{pmatrix} 2\partial_\tau \theta - (\partial_\tau X)^2 & u \\ u & -(\partial_\phi X)^2 \end{pmatrix}, \quad (3.40)$$

$$u = \partial_\phi \theta - \partial_\tau X \partial_\phi X. \quad (3.41)$$

Equations of motion are presented in Hamiltonian form,

$$p = \frac{\partial \mathcal{L}_{\text{NG}}}{\partial \partial_\tau X}, \quad \Pi = \frac{\partial \mathcal{L}_{\text{NG}}}{\partial \partial_\tau \theta}, \quad (3.42)$$

$$\partial_\tau X = -\frac{1}{\Pi} p - u \partial_\phi X, \quad (3.43a)$$

$$\partial_\tau \theta = \frac{1}{2\Pi^2} (p^2 + (\partial_\phi X)^2) - u \partial_\phi \theta, \quad (3.43b)$$

$$\partial_\tau p = -\partial_\phi \left(\frac{1}{\Pi} \partial_\phi X \right) - \partial_\phi (up), \quad (3.43c)$$

$$\partial_\tau \Pi = -\partial_\phi (u\Pi), \quad (3.43d)$$

and there is the constraint

$$p\partial_\phi X + \Pi\partial_\phi\theta = 0. \quad (3.44)$$

There still remains the reparametrization freedom of replacing ϕ by an arbitrary function of τ and ϕ ; this freedom may be used to set $u = 0$, $\Pi = -1$. Consequently, in the fully parametrized equations of motion equation (3.43d) disappears; instead of (3.43a) and (3.43c), we have $\partial_\tau X = p$, $\partial_\tau p = \partial_\phi^2 X$, which imply

$$(\partial_\tau^2 - \partial_\phi^2)X = 0, \quad (3.45a)$$

equation (3.43b) reduces to

$$\partial_\tau\theta = \frac{1}{2}[(\partial_\tau X)^2 + (\partial_\phi X)^2], \quad (3.45b)$$

and the constraint (3.44) requires

$$\partial_\phi\theta = \partial_\tau X\partial_\phi X. \quad (3.45c)$$

Solution to (3.45a) is immediate in terms of two functions F_\pm ,

$$x(\tau, \phi) = F_+(\tau + \phi) + F_-(\tau - \phi) \quad (3.46)$$

and then (3.45b), (3.45c) fix θ :

$$\theta(\tau, \phi) = \int^{\tau+\phi} dz [F'_+(z)]^2 + \int^{\tau-\phi} dz [F'_-(z)]^2. \quad (3.47)$$

This completes the description of a string moving on a plane.

But we need to convert this information into a solution of the Chaplygin gas, and we know from section 2.2.3 that this can be accomplished by a hodographic transformation: instead of X and θ as a function of τ and ϕ , we seek ϕ as a function of τ and X , and this renders θ to be a function of τ and X as well. The density ρ is determined by the Jacobian $|\partial X/\partial\phi|$.

Replace τ by t and X by x and define ϕ to be $f(t, x)$. Then from (3.46) it follows that

$$x = F_+(t + f(t, x)) + F_-(t - f(t, x)). \quad (3.48)$$

This equation may be differentiated with respect to t and x , whereupon one finds

$$\dot{f} = -\frac{F'_+(t + f) + F'_-(t - f)}{F'_+(t + f) - F'_-(t - f)}, \quad (3.49a)$$

$$f' = \frac{1}{F'_+(t + f) - F'_-(t - f)}. \quad (3.49b)$$

Thus the procedure for constructing a Chaplygin gas solution is to choose two functions F_\pm , solve the differential equations (3.49) for f , and then the fluid variables are

$$\theta(t, x) = \int^{t+f(t,x)} [F'_+(z)]^2 dz + \int^{t-f(t,x)} [F'_-(z)]^2 dz, \quad (3.50)$$

$$\frac{\sqrt{2\lambda}}{\rho} = |F'_+(t + \phi) - F'_-(t - \phi)|. \quad (3.51)$$

One may verify directly that (3.50) and (3.51) satisfy the required equations: upon differentiating (3.50) with respect to t and x , we find

$$\begin{aligned} \dot{\theta} &= (F'_+)^2(1 + \dot{f}) + (F'_-)^2(1 - \dot{f}) \\ &= -2F'_+F'_- \end{aligned} \quad (3.52a)$$

$$\begin{aligned}\theta' &= (F'_+)^2 f' - (F'_-)^2 f' \\ &= F'_+ + F'_-.\end{aligned}\tag{3.52b}$$

The second equalities follow with the help of (3.49). From (3.52) one sees that

$$\dot{\theta} + \frac{1}{2}(\theta')^2 = \frac{1}{2}(F'_+ - F'_-)^2 = \frac{\lambda}{\rho^2}\tag{3.53}$$

the last equality being the definition (3.51). Thus the Bernoulli (Euler) equation holds. For the continuity equation, we first find from (3.51) and (3.52)

$$\begin{aligned}\dot{\rho} &= \pm \frac{\partial}{\partial t} \frac{\sqrt{2\lambda}}{F'_+ - F'_-} \\ &= \mp \frac{\sqrt{2\lambda}}{(F'_+ - F'_-)^2} [F''_+(1 + \dot{f}) - F''_-(1 - \dot{f})] \\ &= \pm \frac{2\sqrt{2\lambda}}{(F'_+ - F'_-)^3} (F''_+ F'_- + F''_- F'_+),\end{aligned}\tag{3.54a}$$

$$\begin{aligned}\frac{\partial}{\partial x}(\rho\theta') &= \frac{\partial}{\partial x} \left(\pm \sqrt{2\lambda} \frac{F'_+ + F'_-}{F'_+ - F'_-} \right) \\ &= \mp \frac{\sqrt{2\lambda}}{(F'_+ - F'_-)^2} (F''_+ F'_- + F''_- F'_+) f' \\ &= \mp \frac{2\sqrt{2\lambda}}{(F'_+ - F'_-)^3} (F''_+ F'_- + F''_- F'_+).\end{aligned}\tag{3.54b}$$

The last equalities follow from (3.49); since (3.54a) and (3.54b) sum to zero, the continuity equation holds.

We observe that the differentiated functions F'_\pm are just the Riemann coordinates: from (3.52b) and (3.51) (with the absolute value ignored) we have

$$p \pm \frac{\sqrt{2\lambda}}{\rho} \equiv R_\pm = 2F'_\pm.\tag{3.55}$$

Also it is seen with the help of (3.49) that the Riemann formulation (3.19) of the Chaplygin equations is satisfied by $2F'_\pm$.

The constants of motion (3.17) become proportional to

$$\begin{aligned}I_n^\pm &\propto \int dx \frac{1}{F'_+ - F'_-} [F'_\pm]^n \\ &= \int dx \frac{\partial f}{\partial x} [F'_\pm(t \pm f)]^n \\ &\propto \int dz [F'_\pm(z)]^n.\end{aligned}\tag{3.56}$$

Finally we remark that the solution (3.10), (3.11) corresponds to

$$F_+(z) = -F_-(z) = \pm \frac{\ln z}{2k}.\tag{3.57}$$

There exists a relation between the two functions F and G in (3.26), which encode the Chaplygin gas solution in the linearization approach of section 3.1.2, and the above two functions F_\pm , which do the same job in the Nambu–Goto approach. The relation is that $2F'_+$ is inverse of $2F''$ and $2F'_-$ is inverse of $2G''$, that is,

$$2F''[2F'_+(z)] = z, \quad 2G''[2F'_-(z)] = z.\tag{3.58}$$

4. Supersymmetric fluid mechanics

As explained in section 1, classical fluids describe particles moving collectively. The fluid inherits its mechanical properties, such as energy, momentum and angular momentum from the corresponding underlying particle properties.

One consequence of this is that classical fluids cannot carry intrinsic spin. To be precise, the angular momentum with respect to the centre of mass of a small volume V scales as its mass (which scales as V) times the residual velocity of the fluid about the centre of mass (which scales like ℓ —the linear dimension of V) times the distance to the centre of mass (which also scales like ℓ). Therefore, the intrinsic angular momentum density scales like ℓ^2 , and goes to zero with ℓ .

The inclusion of a spin density in fluids can be achieved in an essentially quantum mechanical formulation by introducing Grassmann (anticommuting) variables in the description, the spin density being represented as a bilinear in the Grassmann variables. This description reveals the possibility of implementing within fluid mechanics supersymmetry transformations, which effectively mix spin and kinematical degrees of freedom. Particular forms of the Hamiltonian, generalizing the classical Chaplygin gas, admit these supersymmetry transformations as an invariance and generate conserved quantities.

Supersymmetry poses severe restrictions in the dynamics. In general, supersymmetric extended objects cannot be formulated in arbitrary dimensions, and this holds true for supersymmetric fluids. It is natural, therefore, that the supersymmetric Chaplygin models are essentially related to, and derive from, higher dimensional supersymmetric membrane actions in a way similar to that already exposed in section 2.2. As such, they also enjoy non-trivial higher dimensional relativistic symmetries which are not apparent from their action.

In the following we shall analyse the case of planar [29] and lineal [30] fluids, which devolve from the motion of membranes or strings in $(3 + 1)$ - or $(2 + 1)$ -dimensional spacetimes, respectively.

Lineal and planar supersymmetric fluid models seem to exhaust the possibilities for the supersymmetric Nambu–Goto/fluid connection. For a higher dimensional generalization, the reduction program would begin with a p -brane in $(p + 2)$ -dimensional spacetime, giving rise to a fluid in $(p + 1)$ -dimensional spacetime. While there are no constraints on p in the purely bosonic case, supersymmetric extensions are greatly constrained: the list of possible ‘fundamental’ super p -branes (i.e., with only scalar supermultiplets in the world volume) contains only the above two cases: $p = 2$ in four dimensions and $p = 1$ in three dimensions [31].

4.1. Supersymmetric fluid in $(2 + 1)$ dimensions

We begin by positing the fluid model. The Chaplygin gas Lagrangian is supplemented by Grassmann variables ψ_a that are Majorana spinors (real, 2-component: $\psi_a^* = \psi_a$, $a = 1, 2$, $(\psi_1\psi_2)^* = \psi_1^*\psi_2^*$).

The associated Lagrange density reads

$$\mathcal{L} = -\rho \left(\dot{\theta} - \frac{1}{2} \psi \dot{\psi} \right) - \frac{1}{2} \rho \left(\nabla \theta - \frac{1}{2} \psi \nabla \psi \right)^2 - \frac{\lambda}{\rho} - \frac{\sqrt{2\lambda}}{2} \psi \alpha \cdot \nabla \psi. \quad (4.1)$$

Here α^i are two ($i = 1, 2$), 2×2 , real symmetric Dirac ‘alpha’ matrices; in terms of Pauli matrices we can take $\alpha^1 = \sigma^1$, $\alpha^2 = \sigma^3$. Note that the matrices satisfy the following relations,

which are needed to verify subsequent formulae,

$$\begin{aligned} \epsilon_{ab}\alpha_{bc}^i &= \epsilon^{ij}\alpha_{ac}^j \\ \alpha_{ab}^i\alpha_{bc}^j &= \delta^{ij}\delta_{ac} - \epsilon^{ij}\epsilon_{ac} \\ \alpha_{ab}^i\alpha_{cd}^i &= \delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc} \end{aligned} \tag{4.2}$$

where ϵ_{ab} is the 2×2 antisymmetric matrix $\epsilon \equiv i\sigma^2$. In equation (4.1) λ is a coupling strength which is assumed to be positive. The Grassmann term enters with coupling $\sqrt{2\lambda}$, which is correlated with the strength of the Chaplygin potential $V(\rho) = \lambda/\rho$ in order to ensure supersymmetry, as we shall show below. It is evident that the velocity should be defined as

$$\mathbf{v} = \nabla\theta - \frac{1}{2}\psi\nabla\psi. \tag{4.3}$$

The Grassmann variables directly give rise to a Clebsch formula for \mathbf{v} , and provide two Monge potentials. The two-dimensional vorticity reads $\omega = \epsilon^{ij}\partial_i v^j = -\frac{1}{2}\epsilon^{ij}\partial_i\psi\partial_j\psi = -\frac{1}{2}\nabla\psi \times \nabla\psi$. The variables $\{\theta, \rho\}$ remain a canonical pair, while the canonical 1-form in (4.1) indicates that the canonically independent Grassmann variables are $\sqrt{\rho}\psi$ so that the antibracket of the ψ is

$$\{\psi_a(\mathbf{r}), \psi_b(\mathbf{r}')\} = -\frac{\delta_{ab}}{\rho(\mathbf{r})}\delta(\mathbf{r} - \mathbf{r}'). \tag{4.4}$$

One verifies that the algebra (1.39)–(1.44) is satisfied, and further, one has

$$\{\theta(\mathbf{r}), \psi(\mathbf{r}')\} = -\frac{1}{2\rho(\mathbf{r})}\psi(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'), \tag{4.5}$$

$$\{\mathbf{v}(\mathbf{r}), \psi(\mathbf{r}')\} = -\frac{\nabla\psi(\mathbf{r})}{\rho(\mathbf{r})}\delta(\mathbf{r} - \mathbf{r}'), \tag{4.6}$$

$$\{\mathcal{P}(\mathbf{r}), \psi(\mathbf{r}')\} = -\nabla\psi(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'). \tag{4.7}$$

The momentum density \mathcal{P} is given by the bosonic formula $\mathcal{P} = \mathbf{v}\rho$, but the Grassmann variables are hidden in \mathbf{v} , by virtue of (4.3).

The equations of motion read

$$\dot{\rho} + \nabla \cdot (\mathbf{v}\rho) = 0, \tag{4.8}$$

$$\dot{\theta} + \mathbf{v} \cdot \nabla\theta = \frac{1}{2}\mathbf{v}^2 + \frac{\lambda}{\rho^2} + \frac{\sqrt{2\lambda}}{2\rho}\psi\boldsymbol{\alpha} \cdot \nabla\psi, \tag{4.9}$$

$$\dot{\psi} + \mathbf{v} \cdot \nabla\psi = \frac{\sqrt{2\lambda}}{\rho}\boldsymbol{\alpha} \cdot \nabla\psi, \tag{4.10}$$

and together with (4.1) they imply

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla\mathbf{v} = \nabla\frac{\lambda}{\rho^2} + \frac{\sqrt{2\lambda}}{\rho}(\nabla\psi)\boldsymbol{\alpha} \cdot \nabla\psi. \tag{4.11}$$

All these equations may be obtained by bracketing with the Hamiltonian

$$H = \int d^2r \left(\frac{1}{2}\rho\mathbf{v}^2 + \frac{\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2}\psi\boldsymbol{\alpha} \cdot \nabla\psi \right) = \int d^2r \mathcal{H} \tag{4.12}$$

when (1.39), (1.43) as well as (4.4)–(4.6) are used.

We record the components of the energy–momentum ‘tensor’, and the continuity equations they satisfy. The energy density $\mathcal{E} = T^{00}$, given by

$$\mathcal{E} = \frac{1}{2}\rho\mathbf{v}^2 + \frac{\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2}\psi\boldsymbol{\alpha}\cdot\nabla\psi = T^{00}, \quad (4.13)$$

satisfies a continuity equation with the energy flux T^{j0} :

$$T^{j0} = \rho v^j \left(\frac{1}{2}\mathbf{v}^2 - \frac{\lambda}{\rho^2} \right) + \frac{\sqrt{2\lambda}}{2}\psi\alpha^j\mathbf{v}\cdot\nabla\psi - \frac{\lambda}{\rho}\psi\partial_j\psi + \frac{\lambda}{\rho}\epsilon^{jk}\psi\epsilon\partial_k\psi \quad (4.14)$$

$$\dot{T}^{00} + \partial_j T^{j0} = 0. \quad (4.15)$$

This ensures that the total energy, that is, the Hamiltonian, is time independent. Conservation of the total momentum

$$\mathbf{P} = \int d^2r \mathcal{P} \quad (4.16)$$

follows from the continuity equation satisfied by the momentum density $\mathcal{P}^i = T^{0i}$ and the momentum flux, that is, the stress tensor T^{ij} :

$$T^{ji} = \rho v^i v^j - \delta^{ij} \left(\frac{2\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2}\psi\boldsymbol{\alpha}\cdot\nabla\psi \right) + \frac{\sqrt{2\lambda}}{2}\psi\alpha^j\partial_i\psi \quad (4.17)$$

$$\dot{T}^{0i} + \partial_j T^{ji} = 0. \quad (4.18)$$

But T^{ij} is not symmetric in its spatial indices, owing to the presence of spin in the problem. However, rotational symmetry makes it possible to effect an ‘improvement’, which modifies the momentum density by a total derivative term, leaving the integrated total momentum unchanged (provided surface terms can be ignored) and rendering the stress tensor symmetric. The improved quantities are

$$\mathcal{P}_I^i = T_I^{0i} = \rho v^i + \frac{1}{8}\epsilon^{ij}\partial_j(\rho\psi\epsilon\psi), \quad (4.19)$$

$$T_I^{ij} = \rho v^i v^j - \delta^{ij} \left(\frac{2\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2}\psi\boldsymbol{\alpha}\cdot\nabla\psi \right) + \frac{\sqrt{2\lambda}}{4}(\psi\alpha^i\partial_j\psi + \psi\alpha^j\partial_i\psi) \\ - \frac{1}{8}\partial_k[(\epsilon^{ki}v^j + \epsilon^{kj}v^i)\rho\psi\epsilon\psi], \quad (4.20)$$

$$\dot{T}_I^{0i} + \partial_j T_I^{ij} = 0. \quad (4.21)$$

It immediately follows from the symmetry of T_I^{ij} that the angular momentum

$$M = \int d^2r \epsilon^{ij} r^i \mathcal{P}_I^j = \int d^2r \rho \epsilon^{ij} r^i v^j + \frac{1}{4} \int d^2r \rho \psi \epsilon \psi \quad (4.22)$$

is conserved. The first term is clearly the orbital part (which still receives a Grassmann contribution through \mathbf{v}), whereas the second, coming from the improvement, is the spin part. Indeed, since $\frac{i}{2}\epsilon = \frac{1}{2}\sigma^2 \equiv \Sigma$, we recognize this as the spin matrix in (2 + 1) dimensions. The extra term in the improved momentum density (4.19), $\frac{1}{8}\epsilon^{ij}\partial_j(\rho\psi\epsilon\psi)$, can then be readily interpreted as an additional localized momentum density, generated by the nonhomogeneity of the spin density. This is analogous to the magnetostatics formula giving the localized current

density \mathbf{j}_m in a magnet in terms of its magnetization \mathbf{m} : $\mathbf{j}_m = \nabla \times \mathbf{m}$. All in all, we are describing a fluid with spin.

Also the total number

$$N = \int d^2r \rho \tag{4.23}$$

is conserved by virtue of the continuity equation (4.8) satisfied by ρ . Finally, the theory is Galileo invariant, as is seen from the conservation of the Galileo boost,

$$\mathbf{B} = t\mathbf{P} - \int d^2r \mathbf{r}\rho \tag{4.24}$$

which follows from (4.8) and (4.16). The generators H , \mathbf{P} , M , \mathbf{B} and N close on the (extended) Galileo group. (The theory is not Lorentz invariant in $(2+1)$ -dimensional spacetime, hence the energy flux T^{j0} does not coincide with the momentum density, improved or not.)

We observe that ρ can be eliminated from (4.1) so that \mathcal{L} involves only θ and ψ . From (4.9) and (4.10) it follows that

$$\rho = \sqrt{\lambda}(\dot{\theta} - \frac{1}{2}\psi\dot{\psi} + \frac{1}{2}\mathbf{v}^2)^{-1/2}. \tag{4.25}$$

Substituting into (4.1) produces the supersymmetric generalization of the Chaplygin gas Lagrange density in (2.17).

$$\mathcal{L} = -2\sqrt{\lambda}\{\sqrt{2\dot{\theta} - \psi\dot{\psi} + (\nabla\theta - \frac{1}{2}\psi\nabla\psi)^2} + \frac{1}{2}\psi\boldsymbol{\alpha} \cdot \nabla\psi\}. \tag{4.26}$$

Note that the coupling strength has disappeared from the dynamical equations, remaining only as a normalization factor for the Lagrangian. Consequently, the above elimination of ρ cannot be carried out in the free case, $\lambda = 0$.

4.2. Supersymmetry

As we stated earlier, this theory possesses supersymmetry. This can be established, first of all, by verifying that the following 2-component supercharges are time-independent Grassmann quantities:

$$Q_a = \int d^2r [\rho \mathbf{v} \cdot (\boldsymbol{\alpha}_{ab} \psi_b) + \sqrt{2\lambda} \psi_a]. \tag{4.27}$$

Taking a time derivative and using the evolution equations (4.8)–(4.11) establishes that $\dot{Q}_a = 0$.

Next, the supersymmetric transformation rule for the dynamical variables is found by constructing a bosonic symmetry generator Q , obtained by contracting the Grassmann charge with a constant Grassmann parameter η^a , $Q = \eta^a Q_a$, and commuting with the dynamical variables. Using the canonical brackets one verifies the following field transformation rules:

$$\delta\rho = \{Q, \rho\} = -\nabla \cdot \rho(\eta\boldsymbol{\alpha}\psi) \tag{4.28}$$

$$\delta\theta = \{Q, \theta\} = -\frac{1}{2}(\eta\boldsymbol{\alpha}\psi) \cdot \nabla\theta - \frac{1}{4}(\eta\boldsymbol{\alpha}\psi) \cdot \psi\nabla\psi + \frac{\sqrt{2\lambda}}{2\rho}\eta\psi \tag{4.29}$$

$$\delta\psi = \{Q, \psi\} = -(\eta\boldsymbol{\alpha}\psi) \cdot \nabla\psi - \mathbf{v} \cdot \boldsymbol{\alpha}\eta - \frac{\sqrt{2\lambda}}{\rho}\eta \tag{4.30}$$

$$\delta\mathbf{v} = \{Q, \mathbf{v}\} = -(\eta\boldsymbol{\alpha}\psi) \cdot \nabla\mathbf{v} + \frac{\sqrt{2\lambda}}{\rho}\eta\nabla\psi. \tag{4.31}$$

Supersymmetry is re-established by determining the variation of the action $\int dt d^2r \mathcal{L}$ consequent to the above field variations: the action is invariant. One then reconstructs

the supercharges (4.27) by Noether's theorem. Finally, upon computing the bracket of two supercharges, one finds

$$\{\eta_1^a Q_a, \eta_2^b Q_b\} = 2(\eta_1 \eta_2) H, \quad (4.32)$$

which again confirms that the charges are time independent:

$$\{H, Q_a\} = 0. \quad (4.33)$$

Additionally a further, kinematical, supersymmetry can be identified. According to the equations of motion, the following two supercharges are also time independent:

$$\bar{Q}_a = \int d^2r \rho \psi_a. \quad (4.34)$$

$\bar{Q} = \bar{\eta}^a \bar{Q}_a$ effects a shift of the Grassmann field:

$$\bar{\delta}\rho = \{\bar{Q}, \rho\} = 0 \quad (4.35)$$

$$\bar{\delta}\theta = \{\bar{Q}, \theta\} = -\frac{1}{2}(\bar{\eta}\psi) \quad (4.36)$$

$$\bar{\delta}\psi = \{\bar{Q}, \psi\} = -\bar{\eta} \quad (4.37)$$

$$\bar{\delta}\mathbf{v} = \{\bar{Q}, \mathbf{v}\} = 0. \quad (4.38)$$

This transformation leaves the Lagrangian invariant, and Noether's theorem reproduces (4.34). The algebra of these charges closes on the total number N ,

$$\{\bar{\eta}_1^a \bar{Q}_a, \bar{\eta}_2^b \bar{Q}_b\} = (\bar{\eta}_1 \bar{\eta}_2) N, \quad (4.39)$$

while the algebra with the generators (2.27) closes on the total momentum, together with a central extension, proportional to volume of space $\Omega = \int d^2r$:

$$\{\bar{\eta}_1^a \bar{Q}_a, \eta^b Q_b\} = (\bar{\eta} \alpha \eta) \cdot \mathbf{P} + \sqrt{2\lambda}(\bar{\eta} \epsilon \eta) \Omega. \quad (4.40)$$

The supercharges Q_a, \bar{Q}_a , together with the Galileo generators $(H, \mathbf{P}, M, \mathbf{B})$ and with N form a superextended Galileo algebra. The additional, non-vanishing brackets are

$$\{M, Q_a\} = \frac{1}{2} \epsilon^{ab} Q_b, \quad (4.41)$$

$$\{M, \bar{Q}_a\} = \frac{1}{2} \epsilon^{ab} \bar{Q}_b, \quad (4.42)$$

$$\{\mathbf{B}, Q_a\} = \alpha_{ab} \bar{Q}_b. \quad (4.43)$$

4.3. Supermembrane connection

The equations for the supersymmetric Chaplygin fluid devolve from a supermembrane Lagrangian, L_M . We shall give two different derivations of this result, which make use of two different parametrizations for the parametrization-invariant membrane action and give rise, respectively, to (4.1) and (4.26). The two derivations follow what has been done in the bosonic case in sections 2.2.1 and 2.2.3.

We work in a light-cone gauge-fixed theory: the supermembrane in four-dimensional spacetime is described by coordinates X^μ ($\mu = 0, 1, 2, 3$), which are decomposed into light-cone components $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^3)$ and transverse components X^i ($i = 1, 2$). These depend on an evolution parameter $\phi^0 \equiv \tau$ and two space-like parameters ϕ^r ($r = 1, 2$). Additionally there are 2-component, real Grassmann spinors ψ , which also depend on τ

and ϕ^r . In the light-cone gauge, X^+ is identified with τ , X^- is renamed θ , and the supermembrane Lagrangian is [32],

$$L_M = \int d^2\phi \mathcal{L}_M = - \int d^2\phi \left\{ \sqrt{G} - \frac{1}{2} \epsilon^{rs} \partial_r \psi \alpha \partial_s \psi \cdot \mathbf{X} \right\}, \quad (4.44)$$

where $G = \det G_{\alpha\beta}$,

$$G_{\alpha\beta} = \begin{pmatrix} G_{00} & G_{0s} \\ G_{r0} & -g_{rs} \end{pmatrix} = \begin{pmatrix} 2\partial_\tau \theta - (\partial_\tau \mathbf{X})^2 - \psi \partial_\tau \psi & u_s \\ u_r & -g_{rs} \end{pmatrix} \quad (4.45)$$

$$\begin{aligned} G &= g\Gamma \\ \Gamma &\equiv 2\partial_\tau \theta - (\partial_\tau \mathbf{X})^2 - \psi \partial_\tau \psi + g^{rs} u_r u_s \\ g_{rs} &\equiv \partial_r \mathbf{X} \cdot \partial_s \mathbf{X}, \quad g = \det g_{rs} \\ u_s &\equiv \partial_s \theta - \frac{1}{2} \psi \partial_s \psi - \partial_\tau \mathbf{X} \cdot \partial_s \mathbf{X}. \end{aligned} \quad (4.46)$$

Here ∂_τ signifies differentiation with respect to the evolution parameter τ , while ∂_r differentiates with respect to the space-like parameters ϕ^r ; g^{rs} is the inverse of g_{rs} , and the two are used to move the (r, s) indices. Note that the dimensionality of the transverse coordinates X^i is the same as of the parameters ϕ^r , namely 2.

4.3.1. Hodographic transformation. To give our first derivation following the procedure in section 2.2.3, we rewrite the Lagrangian in canonical, first-order form, with the help of bosonic canonical momenta defined by

$$\frac{\partial \mathcal{L}_M}{\partial \partial_\tau \mathbf{X}} = \mathbf{p} = -\Pi \partial_\tau \mathbf{X} - \Pi u^r \partial_r \mathbf{X}, \quad (4.47a)$$

$$\frac{\partial \mathcal{L}_M}{\partial \partial_\tau \theta} = \Pi = \sqrt{g/\Gamma}. \quad (4.47b)$$

(The Grassmann variables already enter with first-order derivatives.) The supersymmetric extension of (2.60) then takes the form

$$\begin{aligned} \mathcal{L}_M &= \mathbf{p} \cdot \partial_\tau \mathbf{X} + \Pi \partial_\tau \theta - \frac{1}{2} \Pi \psi \partial_\tau \psi + \frac{1}{2\Pi} (\mathbf{p}^2 + g) + \frac{1}{2} \epsilon^{rs} \partial_r \psi \alpha \partial_s \psi \cdot \mathbf{X} \\ &\quad + u^r \left(\mathbf{p} \cdot \partial_r \mathbf{X} + \Pi \partial_r \theta - \frac{1}{2} \Pi \psi \partial_r \psi \right). \end{aligned} \quad (4.48)$$

In (4.48) u^r serves as a Lagrange multiplier enforcing a subsidiary condition on the canonical variables, and $g = \det g_{rs}$. The equations that follow from (4.48) coincide with the Euler–Lagrange equations for (4.44). The theory still possesses an invariance against redefining the spatial parameters with a τ -dependent function of the parameters. This freedom may be used to set u_τ to zero and fix Π at -1 . Next we introduce the hodographic transformation, as in section 2.2.3, whereby independent–dependent variables are interchanged, namely we view the ϕ^r to be functions of X^i . It then follows that the constraint on (4.48), which with $\Pi = -1$ reads

$$\mathbf{p} \cdot \partial_r \mathbf{X} - \partial_r \theta + \frac{1}{2} \psi \partial_r \psi = 0, \quad (4.49)$$

becomes

$$\partial_r \mathbf{X} \cdot (\mathbf{p} - \nabla \theta + \frac{1}{2} \psi \nabla \psi) = 0. \quad (4.50)$$

Here \mathbf{p} , θ and ψ are viewed as functions of \mathbf{X} , renamed \mathbf{r} , with respect to which the gradient ∇ acts. Also we rename \mathbf{p} as \mathbf{v} , which according to (4.50) is

$$\mathbf{v} = \nabla\theta - \frac{1}{2}\psi\nabla\psi. \quad (4.51)$$

As in section 2.2.3, from the chain rule and the implicit function theorem it follows that

$$\partial_\tau = \partial_t + \partial_\tau \mathbf{X} \cdot \nabla, \quad (4.52)$$

and according to (4.47a) (at $\Pi = -1$, $u^r = 0$) $\partial_\tau \mathbf{X} = \mathbf{p} = \mathbf{v}$. Finally, the measure transforms according to $d^2\phi \rightarrow d^2r \frac{1}{\sqrt{g}}$. Thus the Lagrangian for (4.48) becomes, after setting u^r to zero and Π to -1 ,

$$L_M = \int \frac{d^2r}{\sqrt{g}} \left(\mathbf{v}^2 - \dot{\theta} - \mathbf{v} \cdot \nabla\theta + \frac{1}{2}\psi(\dot{\psi} + \mathbf{v} \cdot \nabla\psi) - \frac{1}{2}(\mathbf{v}^2 + g) - \frac{1}{2}\epsilon^{rs}\psi\alpha^i\partial_j\psi\partial_s X^j\partial_r X^i \right). \quad (4.53a)$$

But $\epsilon^{rs}\partial_s X^j\partial_r X^i = \epsilon^{ij}\det\partial_r X^i = \epsilon^{ij}\sqrt{g}$. After \sqrt{g} is renamed $\sqrt{2\lambda}/\rho$, (4.53a) finally reads

$$L_M = \frac{1}{\sqrt{2\lambda}} \int d^2r \left(-\rho \left(\dot{\theta} - \frac{1}{2}\psi\dot{\psi} \right) - \frac{1}{2}\rho \left(\nabla\theta - \frac{1}{2}\psi\nabla\psi \right)^2 - \frac{\lambda}{\rho} - \frac{\sqrt{2\lambda}}{2}\psi\alpha \times \nabla\psi \right). \quad (4.53b)$$

Upon replacing ψ by $\frac{1}{\sqrt{2}}(1 - \epsilon)\psi$, this is seen to reproduce the Lagrange density (4.1), apart from an overall factor.

4.3.2. Light-cone parametrization. For our second derivation, we return to (4.44)–(4.50) and use the remaining reparametrization freedom to equate the two X^i variables with the two ϕ^r variables, renaming both as r^i . Also τ is renamed as t . This parallels the method in section 2.2.1. Now in (4.44)–(4.46) $g_{rs} = \delta_{rs}$, and $\partial_\tau \mathbf{X} = 0$, so that (4.46) becomes simply

$$G = \Gamma = 2\dot{\theta} - \psi\dot{\psi} + u^2 \quad (4.54)$$

$$\mathbf{u} = \nabla\theta - \frac{1}{2}\psi\nabla\psi. \quad (4.55)$$

Therefore the supermembrane Lagrangian (4.44) reads

$$L_M = - \int d^2r \left\{ \sqrt{2\dot{\theta} - \psi\dot{\psi} + \left(\nabla\theta - \frac{1}{2}\psi\nabla\psi \right)^2} + \frac{1}{2}\psi\alpha \times \nabla\psi \right\}. \quad (4.56)$$

Again a replacement of ψ by $\frac{1}{\sqrt{2}}(1 - \epsilon)\psi$ demonstrates that the integrand coincides with the Lagrange density in (4.26) (apart from a normalization factor).

4.3.3. Further consequences of the supermembrane connection. Supermembrane dynamics is Poincaré invariant in $(3+1)$ -dimensional spacetime. This invariance is hidden by the choice of light-cone parametrization: only the light-cone subgroup of the Poincaré group is left as a manifest invariance. This is just the $(2+1)$ Galileo group generated by H , \mathbf{P} , M , \mathbf{B} and N . (The light-cone subgroup of the Poincaré group is isomorphic to the Galileo group in one lower dimension [23].) The Poincaré generators not included in the above list correspond to Lorentz transformations in the ‘ $-$ ’ direction. We expect therefore that these generators are ‘dynamical’, that is, hidden and unexpected conserved quantities of our supersymmetric Chaplygin gas, similar to the situation with the purely bosonic model.

One verifies that the following quantities,

$$S = tH - \int d^2r \rho\theta \tag{4.57}$$

$$\begin{aligned} \mathbf{G} &= \int d^2r \left(\mathbf{r}\mathcal{H} - \theta\mathcal{P}_I - \frac{1}{8}\psi\alpha\alpha \cdot \mathcal{P}_I\psi \right) \\ &= \int d^2r \left(\mathbf{r}\mathcal{H} - \theta\mathcal{P} - \frac{1}{4}\psi\alpha\alpha \cdot \mathcal{P}\psi \right) \end{aligned} \tag{4.58}$$

are time independent by virtue of the equations of motion (4.8)–(4.11), and they supplement the Galileo generators to form the full (3 + 1) Poincaré algebra, which becomes the super-Poincaré algebra once the supersymmetry is taken into account. Evidently (4.57), (4.58) are the supersymmetric generalizations of (2.14), (2.15).

We see that planar fluid dynamics can be extended to include Grassmann variables, which also enter in a supersymmetry-preserving interaction. Since our construction is based on a supermembrane in (3 + 1)-dimensional spacetime, the fluid model is necessarily a planar Chaplygin gas. In the next section, we shall derive a lineal version of the supersymmetric Chaplygin gas starting from a superstring in (2 + 1)-dimensional spacetime.

4.4. Supersymmetric fluid in (1 + 1) dimensions

The one-dimensional case is in principle simpler since, in one spatial dimension, the canonical structure can be straightforwardly realized. The physical implications of adding Grassmann variables, however, are somewhat limited since there is no vorticity and no spin in one space dimension.

Nevertheless, a supersymmetric version of the lineal Chaplygin gas can be constructed. This is achieved along the same lines as in (2 + 1) dimensions, by considering a superstring moving on a plane and again fixing the parametrization invariance. The construction is analogous to what has already been done in one higher dimension: the Nambu–Goto action for a supermembrane in (3 + 1) dimensions gives rise, in a specific parametrization, to a supersymmetric planar Chaplygin gas.

As we shall demonstrate, the supersymmetric extension enjoys the same integrability properties as the purely bosonic, lineal Chaplygin gas, as a consequence of the complete integrability for the dynamics of the superstring on the plane.

4.4.1. Superstring formulation. We begin with the Nambu–Goto superstring in three-dimensional spacetime,

$$I = - \int d\tau d\sigma (\sqrt{g} - i\epsilon^{ij}\partial_i X^\mu \bar{\psi}\gamma_\mu \partial_j \psi), \tag{4.59}$$

where

$$g = -\det\{\Pi_i^\mu \Pi_j^\nu \eta_{\mu\nu}\}, \tag{4.60}$$

$$\Pi_i^\mu = \partial_i X^\mu - i\bar{\psi}\gamma^\mu \partial_i \psi. \tag{4.61}$$

In these expressions μ, ν are spacetime indices running over 0, 1, 2 and i, j are world-sheet indices denoting τ and σ . We now go to the light-cone gauge where we define $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^2)$. X^+ is identified with the time-like parameter τ , X^- is renamed θ ,

and the remaining transverse component X^1 is renamed x . We can choose a two-dimensional Majorana representation for the γ -matrices:

$$\gamma^0 = \sigma^2, \quad \gamma^1 = -i\sigma^3, \quad \gamma^2 = i\sigma^1,$$

such that ψ is a real, 2-component spinor. A remaining fermionic gauge choice sets

$$\gamma^+ \psi = 0,$$

where $\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^2)$. Thus ψ is further reduced to a real, 1-component Grassmann field. Finally, we define the complex conjugation of a product of Grassmann fields $(\psi_1 \psi_2)^* = \psi_1^* \psi_2^*$ so as to eliminate i from Grassmann bilinears in our final expression. The light-cone gauge-fixed Lagrange density becomes

$$\mathcal{L} = -\sqrt{g}\Gamma + \sqrt{2}\psi\partial_\sigma\psi, \quad (4.62)$$

where

$$g = (\partial_\sigma x)^2, \quad (4.63)$$

$$\Gamma = 2\partial_\tau\theta - (\partial_\tau x)^2 - 2\sqrt{2}\psi\partial_\tau\psi + \frac{u^2}{g}, \quad (4.64)$$

$$u = \partial_\sigma\theta - \partial_\tau x\partial_\sigma x - \sqrt{2}\psi\partial_\tau\psi. \quad (4.65)$$

In the above equations, ∂_σ and ∂_τ denote partial derivatives with respect to the space-like and time-like world-sheet coordinates. The canonical momenta

$$p = \frac{\partial\mathcal{L}}{\partial(\partial_\tau x)} = \sqrt{\frac{g}{\Gamma}} \left(\partial_\tau x + \frac{u}{g}\partial_\sigma x \right), \quad (4.66)$$

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_\tau\theta)} = -\sqrt{\frac{g}{\Gamma}}, \quad (4.67)$$

satisfy the constraint equation

$$p\partial_\sigma x + \Pi\partial_\sigma\theta - \sqrt{2}\Pi\psi\partial_\sigma\psi = 0 \quad (4.68)$$

and can be used to recast \mathcal{L} into the form

$$\begin{aligned} \mathcal{L} = & p\partial_\tau x + \Pi\partial_\tau\theta + \frac{1}{2\Pi}(p^2 + g) + \sqrt{2}\psi\partial_\sigma\psi - \sqrt{2}\Pi\psi\partial_\tau\psi \\ & + u(p\partial_\sigma x + \Pi\partial_\sigma\theta - \sqrt{2}\Pi\psi\partial_\sigma\psi), \end{aligned} \quad (4.69)$$

where u is now a Lagrange multiplier enforcing the constraint. We use the remaining parametrization freedom to fix $u = 0$ and $\Pi = -1$ and perform a hodographic transformation, interchanging independent with dependent variables. The partial derivatives transform by the chain rule,

$$\partial_\sigma = (\partial_\sigma x)\partial_x = \sqrt{g}\partial_x, \quad (4.70)$$

$$\partial_\tau = \partial_t + (\partial_\tau x)\partial_x = \partial_t + v\partial_x, \quad (4.71)$$

and the measure transforms with a factor of $1/\sqrt{g}$. Finally, after renaming \sqrt{g} as $\sqrt{2\lambda}/\rho$, we obtain the Lagrangian for the Chaplygin ‘super’ gas in (1 + 1) dimensions,

$$L = \frac{1}{\sqrt{2\lambda}} \int dx \left\{ -\rho \left(\dot{\theta} - \frac{1}{2}\psi\dot{\psi} \right) - \frac{1}{2}\rho v^2 - \frac{\lambda}{\rho} + \frac{\sqrt{2\lambda}}{2}\psi\psi' \right\}, \quad (4.72)$$

where according to (4.66) and (4.68) (at $u = 0$ and $\Pi = -1$)

$$v = p = \theta' - \frac{1}{2}\psi\psi'. \tag{4.73}$$

We have used ρ and v in anticipation of their role as the fluid density and velocity, and we demonstrate below that they indeed satisfy appropriate equations of motion. For convenience, we have also rescaled ψ everywhere by a factor of $2^{-3/4}$. The Lagrangian (4.72) agrees with the limiting case of the planar fluid in (4.1). We note that as for the planar case, a more straightforward derivation leads to the fluid Lagrangian of (4.72) with ρ integrated out. Specifically, if the parametrization freedom is used directly to equate the space-like and time-like coordinates σ and τ with x and t , we obtain

$$L' = - \int dx \left(\sqrt{2\dot{\theta} - \psi\dot{\psi} + v^2} - \frac{1}{2}\psi\psi' \right), \tag{4.74}$$

where v is defined as in (4.73). This form of the Lagrangian can be obtained from (4.72) after ρ is eliminated using the equations of motion for θ and ψ , shown below.

4.4.2. Supersymmetric Chaplygin gas. (a) Equations of motion. The following equations of motion are obtained by variation of the Lagrangian (4.72):

$$\dot{\rho} + \partial_x(\rho v) = 0 \tag{4.75}$$

$$\dot{\psi} + \left(v + \frac{\sqrt{2\lambda}}{\rho} \right) \psi' = 0 \tag{4.76}$$

$$\dot{\theta} + v\theta' = \frac{1}{2}v^2 + \frac{\lambda}{\rho^2} - \frac{\sqrt{2\lambda}}{2\rho}\psi\psi' \tag{4.77}$$

$$\dot{v} + vv' = \partial_x \left(\frac{\lambda}{\rho^2} \right). \tag{4.78}$$

Naturally, there are only three independent equations of motion as (4.78) is obtained from (4.76), (4.77) and (4.73). Equations (4.75) and (4.78) are seen to be just the continuity and Euler equations for the Chaplygin gas. Note that these do not see the Grassmann variables directly.

We now pass to the Riemann coordinates, which for this system are (velocity \pm speed of sound $\sqrt{2\lambda/\rho}$):

$$R_{\pm} = \left(v \pm \frac{\sqrt{2\lambda}}{\rho} \right). \tag{4.79}$$

In terms of the Riemann coordinates, the equations of motion obtain the form

$$\dot{R}_{\pm} = -R_{\mp}R'_{\pm}, \tag{4.80}$$

$$\dot{\psi} = -R_+\psi', \tag{4.81}$$

$$\dot{\theta} = -\frac{1}{2}R_+R_- - \frac{1}{2}R_+\psi\psi'. \tag{4.82}$$

The equations in (4.80) contain the continuity and Euler equations and are known to be integrable. It is readily verified that equation (4.81) for ψ is solved by any function of R_- ,

$$\psi = \Psi(R_-), \tag{4.83}$$

and hence the fluid model is completely integrable. That this is the case should come as no surprise considering that we began with an integrable world-sheet theory.

At this point it may seem curiously asymmetric that equation (4.81) for the Grassmann field should contain the R_+ Riemann coordinate and not the R_- companion coordinate. In fact, the reverse would have been the case if the sign of the $\sqrt{2\lambda}$ term in (4.72) had been opposite. The entire model is consistent with this substitution, which is just the choice of identifying \sqrt{g} with plus or minus the speed of sound $\sqrt{2\lambda}/\rho$.

The energy–momentum tensor is constructed from (4.72), and its components are

$$T^{00} = \mathcal{H} = \frac{1}{2}\rho v^2 + \frac{\lambda}{\rho} - \frac{\sqrt{2\lambda}}{2}\psi\psi', \quad (4.84)$$

$$T^{01} = \mathcal{P} = \rho v, \quad (4.85)$$

$$T^{10} = \frac{\rho v}{2}R_+R_- - \frac{\sqrt{2\lambda}}{2}R_+\psi\psi', \quad (4.86)$$

$$T^{11} = \rho R_+R_-. \quad (4.87)$$

The expected conserved quantities of the system, the generators of the Galileo group, are verified to be time independent using the equations of motion. We have

$$N = \int dx \rho, \quad (4.88)$$

$$P = \int dx \rho v, \quad (4.89)$$

$$H = \int dx \left(\frac{1}{2}\rho v^2 + \frac{\lambda}{\rho} - \frac{\sqrt{2\lambda}}{2}\psi\psi' \right), \quad (4.90)$$

$$B = \int dx \rho(x - vt) = \int dx x\rho - tP. \quad (4.91)$$

Although some generators look purely bosonic, there are still Grassmann fields hidden in v according to its definition (4.73).

In going to Riemann coordinates, we can observe a ladder of conserved charges of the form

$$I_n^\pm = \int dx \rho R_\pm^n. \quad (4.92)$$

The first few values of n above give

$$I_0^\pm = N, \quad (4.93)$$

$$I_1^\pm = P \pm \sqrt{2\lambda}\Omega, \quad (4.94)$$

$$I_2^+ = 2H, \quad (4.95)$$

where Ω is used to denote the length of space $\int dx$. (Note that I_2^- would correspond to the Hamiltonian of the theory with $\sqrt{2\lambda}$ replaced by its negative.)

In section 4.2 we identified two different supersymmetry generators, which correspond in one space dimension to the time-independent quantities

$$\tilde{Q} = \int dx \rho\psi, \quad (4.96)$$

$$Q = \int dx \rho \left(v - \frac{\sqrt{2\lambda}}{\rho} \right) \psi. \tag{4.97}$$

These are again special cases ($n = 0$ and $n = 1$) of a ladder of conserved supercharges described by

$$Q_n = \int dx \rho R_-^n \psi. \tag{4.98}$$

We see that the supercharges evaluated on the solution (4.83) reproduce the form of the bosonic charges (3.17).

Let us observe that there exist further bosonic and fermionic conserved charges. For example, one may verify that the bosonic charges

$$\int dx \rho R_{\pm}^n \left(\frac{R'_{\pm}}{\rho} \right)^m \tag{4.99}$$

$$\int dx \rho R_-^n \left(\frac{\psi \psi'}{\rho} \right) \tag{4.100}$$

are conserved, as are the fermionic charges

$$\int dx \rho R_-^n \left(\frac{\psi'}{\rho} \right). \tag{4.101}$$

Conserved expressions involving higher derivatives may also be constructed. The conservation of these quantities is easily understood when the string world-sheet variables are used. Then the above are written as $\int d\sigma R_{\pm}^n (\partial_{\sigma} R_{\pm})^m$, $\int d\sigma R_-^n (\psi \partial_{\sigma} \psi)$ and $\int d\sigma R_-^n (\partial_{\sigma} \psi)$, respectively. Furthermore when R_{\pm} are evaluated on solutions, they become functions of $\tau \pm \sigma$ (see section 3.3), so that integration over σ extinguishes the τ dependence, leaving constant quantities.

(b) *Canonical structure.* The equations of motion (4.75)–(4.77) can also be obtained by Poisson bracketing with the Hamiltonian (4.84) if the following canonical brackets are postulated,

$$\{\theta(x), \rho(y)\} = \delta(x - y) \tag{4.102}$$

$$\{\theta(x), \psi(y)\} = -\frac{\psi}{2\rho} \delta(x - y) \tag{4.103}$$

$$\{\psi(x), \psi(y)\} = -\frac{1}{\rho} \delta(x - y) \tag{4.104}$$

where the last bracket, containing Grassmann arguments on both sides is understood to be the antibracket. With these one verifies that the conserved charges in (4.88)–(4.91) generate the appropriate Galileo symmetry transformations on the dynamical variables ρ, θ and ψ . Correspondingly the supercharges (4.96), (4.97) generate super transformations,

$$\tilde{\delta}\rho = 0 \qquad \delta\rho = -\eta \partial_x (\rho \psi) \tag{4.105}$$

$$\tilde{\delta}\theta = -\frac{1}{2} \eta \psi \qquad \delta\theta = -\frac{1}{2} \eta R_+ \psi \tag{4.106}$$

$$\tilde{\delta}\psi = -\eta \qquad \delta\psi = -\eta \psi \psi' - \eta R_- \tag{4.107}$$

which leave the Lagrangian (4.72) invariant. The algebra of the bosonic generators reproduces the algebra of the (extended) Galileo group. The algebra of the supercharges is

$$\{\bar{\eta}Q, \eta Q\} = 2\bar{\eta}\eta H, \quad (4.108)$$

$$\{\bar{\eta}\tilde{Q}, \eta\tilde{Q}\} = \bar{\eta}\eta N, \quad (4.109)$$

$$\{\bar{\eta}\tilde{Q}, \eta Q\} = \bar{\eta}\eta(P - \sqrt{2\lambda}\Omega), \quad (4.110)$$

$$\{B, Q\} = \tilde{Q}. \quad (4.111)$$

(c) *Additional symmetries of the fluid model.* As mentioned above, since the fluid model descends from the superstring, it should possess an enhanced symmetry beyond the Galileo symmetry in (1 + 1) dimensions. In fact, the following conserved charges effecting time rescaling and spacetime mixing are also established:

$$D = tH - \int dx \rho\theta, \quad (4.112)$$

$$G = \int dx (x\mathcal{H} - \theta\mathcal{P}), \quad (4.113)$$

The Galileo generators supplemented by D and G together satisfy the Lie algebra of the (2 + 1)-dimensional Poincaré group, with N , P and H corresponding to the three translations and with B , D and G forming the (2 + 1)-dimensional Lorentz group $SO(2, 1)$,

$$\{B, D\} = B, \quad \{G, B\} = D, \quad \{D, G\} = G, \quad (4.114)$$

with Casimir

$$C = B \circ G + G \circ B + D \circ D. \quad (4.115)$$

Adjoining the supercharges results in the super-Poincaré algebra of (2 + 1) dimensions. The Lorentz charges do not belong to the infinite towers of constants of motion mentioned earlier. Rather, they act as raising and lowering operators. One verifies for the Q_n and I_n^+ :

$$\begin{aligned} \{B, I_n^+\} &= -nI_{n-1}^+, & \{D, I_n^+\} &= (n-1)I_n^+, & \{G, I_n^+\} &= \left(\frac{n}{2} - 1\right)I_{n+1}^+, \\ \{B, Q_n\} &= -nQ_{n-1}, & \{D, Q_n\} &= \left(n - \frac{1}{2}\right)Q_n, & \{G, Q_n\} &= \left(\frac{n}{2} - \frac{1}{2}\right)Q_{n+1}. \end{aligned} \quad (4.116)$$

(Note that the $\{B, I_2^+\}$ bracket coincides with $\{B, 2H\}$, which should equal $-2P$ according to the Galileo algebra. But the above result, namely $-2I_1^+$, gives $-2(P - \sqrt{2\lambda}\Omega)$. This central addition arises from a term of the form $\int dx dy \sqrt{2\lambda}x\delta'(x-y)$, whose value is ambiguous, depending on the order of integration.) The brackets with the I_n^- do not close, but the I_n^- can be modified by the addition of another tower of constant quantities, namely those of (4.100):

$$\tilde{I}_n^- = I_n^- - \sqrt{2\lambda}n(n-1) \int dx R_-^{n-2} \psi \psi'. \quad (4.117)$$

The modified constants obey the same algebra as I_n^+ ,

$$\{B, \tilde{I}_n^-\} = -n\tilde{I}_{n-1}^-, \quad \{D, \tilde{I}_n^-\} = (n-1)\tilde{I}_n^-, \quad \{G, \tilde{I}_n^-\} = \left(\frac{n}{2} - 1\right)\tilde{I}_{n+1}^-. \quad (4.118)$$

Evidently I_n^+ , \tilde{I}_n^- and Q_n provide irreducible, infinite-dimensional representations for $SO(2, 1)$, with the Casimir, in adjoint action, taking the form $l(l+1)$ and $l = 1$ for I_n^+ , \tilde{I}_n^- and $l = 1/2$ for Q_n .

We enquire about the algebra of the towers of extended charges I_n^+ , \tilde{I}_n^- and Q_n . While some (bosonic) brackets vanish, others provide new constants of motion like those in (4.99)–(4.101) and their generalizations with more derivatives. Thus it appears that one is dealing with an open (super)algebra.

A final comment. We have presented supersymmetric versions of fluid dynamical models in two and one space dimension. These models enjoy supersymmetry as well as extra ‘dynamical’ symmetries tracing back to their origin in higher dimensional supermembrane models. Other investigations of supersymmetric fluids are reported in [33].

There remain some obvious open questions. One is what other fluid interactions can be obtained from the rich factory of (super)branes. For example, string theory D -branes have gauge fields living on them. Such gauge fields would presumably remain in passing to a fluid model and may thus provide a model of (super)magnetohydrodynamics from D -branes.

Another question is the construction and properties of fluid models with Grassmann variables in arbitrary dimensions. Such models would not descend from supermembrane models and, consequently, would not enjoy supersymmetry or other extended symmetries. Note, nevertheless, that Grassmann Monge potentials ψ can be used even in the absence of supersymmetry. For example, the above models with the last explicitly fermion-dependent term omitted, possess a conventional, bosonic Hamiltonian without supersymmetry, while the Grassmann variables are hidden in \mathbf{v} and occur only in the canonical 1-form. These models would describe fluids with fundamental spin degrees of freedom and it would be worthwhile to explore their physical properties in this description.

5. Non-Abelian fluids

5.1. Introduction

In this section we generalize fluid dynamics to systems with non-Abelian charges. We begin with some comments on the physical contexts in which such a generalization might be useful and the scope and limitations of possible approaches to the problem.

The quintessential example of a physical system with non-Abelian charges is the quark–gluon plasma. High-energy collisions of heavy nuclei can produce a plasma state of quarks and gluons. This new state of matter has recently been of great interest both theoretically and in experiments at the RHIC facility and at CERN. In fact, there is growing evidence that such a state has already been achieved at RHIC [34]. In attempting a theoretical description, there are basically two approaches that we can use. Since the plasma is at high temperatures, one can argue that the average energy per particle is high enough to justify the use of perturbative *quantum chromodynamics* by virtue of asymptotic freedom. However, it is known that because of the infrared divergences various resummations, such as summing hard thermal loop contributions, have to be done before a perturbative expansion with control of the infrared degrees of freedom can be set up [35]. One has to address also the question of chromomagnetic screening, because unlike the Abelian plasma, there can be spatial screening of magnetic-type interactions [36]. The expected end result is then a good description valid at high temperatures and for plasma states that are not too far from equilibrium, since one is perturbing around the equilibrium state. An alternative approach, which may be more suitable for nondilute plasmas or for situations far from equilibrium, would be to use a fluid mechanical description.

We begin by observing that many of the general comments given in the introduction, on deriving fluid mechanics from an underlying particle theory by statistical averages, will apply in the non-Abelian context as well. Specifically for the quark–gluon plasma, some work

along these lines was done many years ago using single-particle kinetic equations [37]. The one-particle kinetic equation takes the form

$$P^\mu \left[\frac{\partial}{\partial X^\mu} + g Q_a F_{\mu\nu}^a \frac{\partial}{\partial P_\nu} + g f_{abc} A_\mu^b Q_c \frac{\partial}{\partial Q_a} \right] f(X, P, Q) = \mathcal{C}(f). \quad (5.1)$$

Here $f(X, P, Q)$ is the one-particle distribution function and \mathcal{C} is the collision integral term taking account of scatterings of the particles. A_μ^a and $F_{\mu\nu}^a$ are the potential and field for a non-Abelian theory based on a gauge group with structure constants f_{abc} . Here Q_a represents the classical colour charge of the particle.

It may be interesting to note that, for a collisionless plasma, i.e., with $\mathcal{C} = 0$, the Boltzmann equation (5.1) is the equation for the distribution function for single particles obeying the standard classical equations of motion for non-Abelian particles—the so-called Wong equations [38]—which are

$$\begin{aligned} m \frac{dX^\mu}{d\tau} &= P^\mu, \\ m \frac{dP_\mu}{d\tau} &= g Q_a F_{\mu\nu}^a P^\nu, \\ m \frac{dQ_a}{d\tau} &= -g f_{abc} (P^\mu A_\mu^b) Q_c. \end{aligned} \quad (5.2)$$

As we shall see shortly, the motion of the colour degrees of freedom can also be described in a phase space way; the appropriate space is the Lie group modulo the maximal torus.

The Boltzmann equation (5.1) is invariant under gauge transformations in the sense that if $f(X, P, Q)$ solves (5.1), then so does $f(X, P, U^{-1}QU)$. As in the Abelian case, one may, in a dilute system, seek a perturbative solution of the form $f = f^{(0)} + g f^{(1)} + \dots$, where $f^{(0)} = n_p$ is the equilibrium distribution; the perturbative corrections can then give the transport coefficients and fluid equations of motion [37]. Such an approach suffers from the same limitations mentioned in the introduction, namely, that it can only be justified for dilute systems near equilibrium and in a semiclassical approximation. However, the fact that the equations for the Abelian fluid have a fairly large regime of validity, despite the fact that it can be derived from an underlying particle theory in a limited context, then prompts us to ask for an *a priori* derivation of a non-Abelian fluid mechanics, which incorporates the non-Abelian degrees of freedom, coupling to a non-Abelian gauge field, etc. This theory may be valid for dense, nonperturbative and nondilute systems. This is the goal of this section. In proceeding with the development of such a theory, it is useful to keep in mind some guidelines or desirable features. First of all, a canonical or symplectic formulation (at least in the conservative limit) is important for quantization, so we should aim for this. At the same time, the analysis based on the kinetic equations still remains useful to us as a guide for arriving at the equations of interest. Our analysis is based on the paper in [39].

5.2. Non-Abelian Euler variables

A possible form for the non-Abelian Euler fluid variables may be inferred from the single-particle equations of motion (5.2) by a procedure analogous to what was in section 1.1 for the Abelian case.

The single-particle non-Abelian current is defined in terms of the Lagrange variables X^μ and Q_a ,

$$J_a^\mu(t, \mathbf{r}) = \int d\tau Q_a(\tau) \frac{dX^\mu(\tau)}{d\tau} \delta(X^0(\tau) - ct) \delta(\mathbf{X}(\tau) - \mathbf{r}). \quad (5.3)$$

With the parametrization $X^0(\tau) = c\tau$,

$$\rho_a(t, \mathbf{r}) = Q_a(t)\delta(\mathbf{X}(t) - \mathbf{r}), \tag{5.4}$$

$$\mathbf{J}_a(t, \mathbf{r}) = Q_a(t)\dot{\mathbf{X}}(t)\delta(\mathbf{X}(t) - \mathbf{r}). \tag{5.5}$$

This generalizes (1.5) and (1.9) by inclusion of the dynamical non-Abelian charge Q_a which satisfies (5.2), so that J_a^μ is covariantly conserved:

$$(D_\mu J^\mu)_a \equiv \partial_\mu J_a^\mu + f_{abc}A_\mu^b J_c^\mu = 0. \tag{5.6}$$

(In the present case, the current is defined without the mass factor and it is normalized by Q_a .) Passage to the fluid description is achieved as in section 1.1. In the many-particle case, dynamical quantities are decorated with the particle label n , as in (1.11)–(1.13), which is summed in the definition of the current. Then in the continuum fluid limit, n is replaced by the continuous variable \mathbf{x} and the charge and current densities read

$$\rho_a(t, \mathbf{r}) = \int d^3x Q_a(t, \mathbf{x})\delta(\mathbf{X}(t, \mathbf{x}) - \mathbf{r}), \tag{5.7}$$

$$\mathbf{J}_a(t, \mathbf{r}) = \int d^3x Q_a(t, \mathbf{x})\dot{\mathbf{X}}(t, \mathbf{x})\delta(\mathbf{X}(t, \mathbf{x}) - \mathbf{r}), \tag{5.8}$$

with $Q_a(t, \mathbf{x})$ satisfying

$$\dot{Q}_a(t, \mathbf{x}) + f_{abc}[cA_0^b(t, \mathbf{X}(t, \mathbf{x})) + \dot{\mathbf{X}}(t, \mathbf{x}) \cdot \mathbf{A}^b(t, \mathbf{X}(t, \mathbf{x}))]Q_c(t, \mathbf{x}) = 0. \tag{5.9}$$

(Note that replacing a discrete sum by an integral over \mathbf{x} forces $Q_a(t, \mathbf{x})$ to be a charge density.)

Observe that, just as in the Abelian case discussed in section 1.1, the \mathbf{x} -integration evaluates \mathbf{x} at $\chi(t, \mathbf{r})$, the inverse of \mathbf{X} , and the Jacobian factor $|\det \partial X^i / \partial x^j|_{\mathbf{x}=\chi}$ is just the Abelian charge density ρ (see (1.21), (1.22)). Thus the non-Abelian quantities factorize:

$$\rho_a(t, \mathbf{r}) = Q_a(t, \mathbf{r})\rho(t, \mathbf{r}) \tag{5.10}$$

$$\mathbf{J}_a(t, \mathbf{r}) = Q_a(t, \mathbf{r})\rho(t, \mathbf{r})\mathbf{v}(t, \mathbf{r}). \tag{5.11}$$

Equivalently,

$$J_a^\mu(t, \mathbf{r}) = Q_a(t, \mathbf{r})j^\mu(t, \mathbf{r}), \tag{5.12}$$

where

$$Q_a(t, \mathbf{r}) = Q_a(t, \mathbf{x})|_{\mathbf{x}=\chi}, \tag{5.13}$$

$$\rho(t, \mathbf{r})Q_a(t, \mathbf{r}) = \int dx Q_a(t, \mathbf{x})\delta(\mathbf{X}(t, \mathbf{x}) - \mathbf{r}). \tag{5.14}$$

As a consequence of its definition, the Abelian current factor $j^\mu = (c\rho, \mathbf{v}\rho)$ satisfies its own continuity equation (1.19). Moreover, differentiating (5.14) with respect to time and using (1.19) and (5.9) results in an equation for \dot{Q}_a ,

$$\dot{Q}_a(t, \mathbf{r}) + \mathbf{v}(t, \mathbf{r}) \cdot \nabla Q_a(t, \mathbf{r}) = -f_{abc}[cA_0^b(t, \mathbf{r}) + \mathbf{v}(t, \mathbf{r}) \cdot \mathbf{A}^b(t, \mathbf{r})]Q_c(t, \mathbf{r}) \tag{5.15}$$

which can also be written as

$$j^\mu(D_\mu Q)_a = 0. \tag{5.16}$$

This is analogous to the Abelian equations (1.77). Equations (5.15) and (5.16) can be understood from the fact that the covariantly conserved current (5.3), (5.6) factorizes according to (5.12) into a group variable Q_a and a conserved Abelian current j^μ . Consistency of (1.19),

(5.6) and (5.12) then enforces (5.16). Evidently, this is a generalization of the particle Wong equation (5.2); therefore we shall refer to it as the fluid Wong equation.

We recognize that the formulae (5.7) and (5.8) are the non-Abelian version of the Lagrange variable–Euler variable correspondence (see (1.6) and (1.7)). Also, (5.9) is the field generalization of the particle Wong equation, presented in the Lagrange formalism, and (5.15) and (5.16) are the Euler version of the same. The decomposition of the non-Abelian current in (5.12) is the non-Abelian version of the Eckart decomposition (1.102). Indeed, (5.12) may be further factored as in (1.102):

$$J_a^\mu(t, \mathbf{r}) = Q_a(t, \mathbf{r})n(t, \mathbf{r})u^\mu(t, \mathbf{r}). \quad (5.17)$$

In the remainder of section 5, we are guided in our construction of a dynamical model for non-Abelian fluid mechanics and ‘colour’ hydrodynamics by the above properties of the non-Abelian current, which follow from the very general arguments, based on a particle picture for the substratum of a fluid. In appendix G, we present a different model, based on a field-theoretic fluid substratum.

5.3. Constructing the action

The equations of motion for the non-Abelian fluid in the Euler formulation include the kinematical equations: continuity (5.6) and Wong (5.16) that are satisfied by the non-Abelian current, which is factorized as in (5.12)–(5.17). Still needed is the Euler force equation, analogous to (1.110), which specifies the dynamics. We present this by constructing an action whose variation reproduces the kinematical equations and gives a model for the dynamical equation.

The algebra underlying the non-Abelian theory is realized with anti-Hermitian generators T_a satisfying

$$[T^a, T^b] = f_{abc}T^c, \quad (5.18)$$

and normalized by

$$\text{Tr } T^a T^b = -\frac{1}{2}\delta^{ab}. \quad (5.19)$$

In a canonical particle theory we expect that the algebra (5.18) is reproduced by Poisson brackets for corresponding symmetry generators. In a field theory, we expect to find a copy of (5.18) at each point in space, leading to the Poisson brackets

$$\{\rho_a(\mathbf{r}), \rho_b(\mathbf{r}')\} = f_{abc}\rho_c(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'), \quad (5.20)$$

which generalize (1.39). (A common time argument is suppressed.) Upon quantization the brackets become commutators and acquire the factor i/\hbar . (Schwinger terms do not spoil the quantum algebra unless there are anomalies in the gauge symmetry [40]. Of course, we assume that the theory is anomaly free.)

The action which leads to the commutation rules (5.20) is the field theoretic version of the Kirillov–Kostant form, which for a particle (not a field) reads

$$I_{\text{KK}} = 2n \int dt \text{Tr } T^3 g^{-1} \dot{g}, \quad (5.21)$$

where n is a normalization constant and we have taken the group $SU(2)$ as a concrete example: $g \in SU(2)$, $g = \exp(T^a \varphi_a)$, $T^a = \sigma^a/2i$ and σ^a are Pauli matrices. The action (5.21) is invariant under $g \rightarrow g \exp T^3 \varphi$, modulo surface terms. Therefore, the theory governed by (5.21) is defined on the two-dimensional sphere $SU(2)/U(1) = S^2$. The observables are given by $q_a = -2 \text{Tr}(g T^3 g^{-1} T^a)$, one can show that they obey the bracket rules $\{q_a, q_b\} = \varepsilon_{abc} q_c$,

which is the single-particle version of (5.20). Upon quantization, the quantum Hilbert space will consist of one unitary irreducible representation of the group $SU(2)$ with the highest weight or j -value given by $j = \frac{1}{2}n$. The symplectic 2-form associated with (5.21) is the field of a magnetic monopole on the 2-sphere $S^2 = SU(2)/U(1)$. The Dirac quantization rule requires that n be an integer, consistent with j being half-integral. Interpreting the single irreducible representation of the group as representing the charge degrees of freedom, the action (5.21) describes a single particle with $SU(2)$ non-Abelian charges. One can in fact use (5.21) as part of an action for the Wong equations of motion [8, 41].

More generally, consider a Lie group G with H denoting its Cartan subgroup (or maximal torus). The required generalized action is given by

$$I_0 = \sum_{s=1}^r n_s \int dt \text{Tr} K_{(s)} g^{-1} \dot{g} \tag{5.22}$$

where $g \in G$ and n_s are the highest weights defining a unitary irreducible representation of G , $K_{(s)}$ are the diagonal generators of the Cartan subalgebra H of G . The summation in (5.22) extends up to the rank r of the algebra, but some of the n could vanish. The action (5.22) is invariant under $g \rightarrow gh, h \in H$ and time independent, so that the phase space is G/H , which is known to be a Kähler (and symplectic) space. Quantization leads to a finite-dimensional Hilbert space which carries a unitary irreducible representation of G labelled by the highest weights n_s .

Given this structure, we see that field-theoretic generalization, which would give rise to (5.20), appears as $\int dt dr \sum_{s=1}^r n_{(s)} \text{Tr}(K_{(s)} g^{-1} \dot{g})$, where now n_s and g depend on \mathbf{r} , while the $K_{(s)}$ remain as constant elements of the Cartan subalgebra of the group. Thus we take as the Lagrange density for our non-Abelian fluid dynamics the formula

$$\mathcal{L} = \sum_{s=1}^r j_{(s)}^\mu 2 \text{Tr} K_{(s)} g^{-1} D_\mu g - f(n_{(1)}, n_{(2)}, \dots, n_{(r)}) + \mathcal{L}_{\text{gauge}}. \tag{5.23}$$

Here $j_{(s)}^\mu$ are a set of Abelian currents; they may be taken to be in the Eckart form $j_{(s)}^\mu = n_{(s)} u_{(s)}^\mu$, where u^μ are 4-velocity vectors, with $u_{(s)}^\mu u_{(s)\mu} = 1$. As far as the variational problem of this action is concerned, we regard $n_{(s)}$ as given by $j_{(s)}^\mu$ via $n_{(s)} = \sqrt{j_{(s)}^\mu j_{\mu(s)}}$. The space and time components of the currents are given by $j_{(s)}^\mu = (c\rho_{(s)}, \rho_{(s)} \mathbf{v}_{(s)})$, $\rho_{(s)} = n_{(s)} u_{(s)}^0$. In equation (5.23) $n_{(s)}$ are the invariant densities for the diagonal directions of the Lie algebra.

Comparison with the usual form of the action for an Abelian fluid shows that what we have obtained is the non-Abelian analogue of the irrotational part of the flow. In the Abelian case, and without the gauge field coupling, equation (5.23) entails a single contribution, $s = 1$, and $2 \text{Tr}(K_1 g^{-1} \partial_\mu g) = -\partial_\mu \theta$, with vanishing vorticity. In the non-Abelian case, the vorticity is still non-vanishing. One can easily generalize (5.23) to include the other Gaussian components of the Clebsch parametrized vector which couples to $j_{(s)}^\mu$. This gives the Lagrangian

$$\mathcal{L} = \sum_{s=1}^r j_{(s)}^\mu \{2 \text{Tr} K_{(s)} g^{-1} D_\mu g + a_{\mu(s)}\} - f(n_{(1)}, n_{(2)}, \dots, n_{(r)}) + \mathcal{L}_{\text{gauge}} \tag{5.24}$$

where $a_{\mu(s)}$ is given by

$$a_{\mu(s)} = \alpha_{(s)} \partial_\mu \beta_{(s)}. \tag{5.25}$$

For the rank-1 group $SU(2)$, with its single Cartan element, the s -sum in (5.23) is exhausted by a single element. We see that the $SU(2)$ fluid has one component. For higher rank groups, the non-Abelian fluid can have up to r components, but fewer—indeed even just

a single flow—are possible when some of the densities $n_{(s)}$ vanish. Mathematically, single-component fluids are the simplest, but physically it is unclear what kinematical regimes of a quark–gluon plasma, for example, would admit or even require such a reduction in flows.

The covariant derivative in (5.23), namely,

$$D_\mu g = \partial_\mu g + A_\mu g, \quad (5.26)$$

involves a dynamical non-Abelian gauge potential $A_\mu = A_\mu^a T^a$ whose dynamics is provided by $\mathcal{L}_{\text{gauge}}$. The first term in \mathcal{L} contains the canonical 1-form for our theory and determines the symplectic structure and the canonical brackets. We have added the Hamiltonian density part, the function $f(n_{(1)}, n_{(2)}, \dots, n_{(r)})$ which describes the fluid dynamics. The theory is invariant under gauge transformations with group element U

$$g \rightarrow U^{-1}g, \quad A_\mu \rightarrow U^{-1}(A_\mu + \partial_\mu)U, \quad (5.27)$$

where $U \in G$. In appendix D we show that the canonical structure of our theory leads to the charge density algebra (5.20).

5.4. Equations of motion

In working out the equations of motion and other consequences of this theory, it is instructive to consider first the simpler case of a single flow. This is obtained for $G = SU(2)$, but could also occur in higher rank groups.

5.4.1. Single-component flow. The single flow Lagrangian takes the form

$$\mathcal{L} = j^\mu 2 \text{Tr} K g^{-1} D_\mu g - f(n) + \mathcal{L}_{\text{gauge}}, \quad (5.28)$$

where $K = \sigma_3/2i$. Now we have a single Abelian current j^μ which can be decomposed as

$$j^\mu = (c\rho, \mathbf{v}\rho) = nu^\mu, \quad u^\mu u_\mu = 1. \quad (5.29)$$

The current J_a^μ to which A_μ^a couples is easily worked out from (5.28). Upon defining

$$gKg^{-1} \equiv Q = Q_a T^a, \quad (5.30)$$

we find

$$J_a^\mu = (c\rho_a, \mathbf{J}_a) = Q_a j^\mu = Q_a n u^\mu, \quad \rho_a = \rho Q_a. \quad (5.31)$$

Q_a may be thought of as the charge of a single particle, with $\rho_a = \rho Q_a$ as the non-Abelian charge density at the point \mathbf{r} . Note that this is of the Eckart form where the currents are given by charge densities multiplied by the velocity vector. This agrees with the general discussion in section 5.2, where we showed that a particle-based model for the fluid leads to the Eckart forms (5.10), (5.11).

The gauge invariance of the Lagrangian (5.28) shows that the current $J^\mu \equiv J_a^\mu T^a$ must be covariantly conserved, i.e.,

$$(D_\mu J^\mu)_a = \partial_\mu J_a^\mu + f_{abc} A_\mu^b J_c^\mu = 0. \quad (5.32)$$

Further the current j^μ satisfies an ordinary conservation law,

$$\partial_\mu j^\mu = 0. \quad (5.33)$$

In appendix E, we show that both conservation laws are a consequence of invariance of the action with respect to variations of the group element g : arbitrary left variations of g lead to covariant conservation of J_a^μ (5.32) while the particular variation $\delta g = gK\lambda$, with λ an arbitrary function of spacetime, ensures that j^μ is conserved as in (5.33). We shall see later

that these can be interpreted in terms of the Wong equations. Note that these conservation laws lead to the fluid Wong equation

$$j^\mu (D_\mu Q)_a = 0 \tag{5.34}$$

as has already been noted in (5.16).

It remains to derive the Euler equation. This is accomplished by varying j^μ ; stationary variation requires

$$2 \operatorname{Tr}[Q(D_\mu g)g^{-1}] = 2 \operatorname{Tr}(Kg^{-1}D_\mu g) = \frac{u_\mu}{c^2} f'(n), \tag{5.35}$$

which we call the non-Abelian Bernoulli equation. The Euler equation then follows, as in the Abelian case, by taking the curl:

$$\partial_\mu (2 \operatorname{Tr} Q(D_\nu g)g^{-1}) - \partial_\nu (2 \operatorname{Tr} Q(D_\mu g)g^{-1}) = \partial_\mu \left(\frac{u_\nu}{c^2} f'(n) \right) - \partial_\nu \left(\frac{u_\mu}{c^2} f'(n) \right). \tag{5.36}$$

In appendix F, we show that manipulating the left-hand side allows rewriting (5.36) as

$$2 \operatorname{Tr}(D_\mu Q)(D_\nu g)g^{-1} + 2 \operatorname{Tr} Q F_{\mu\nu} = \partial_\mu \left(\frac{u_\nu}{c^2} f'(n) \right) - \partial_\nu \left(\frac{u_\mu}{c^2} f'(n) \right). \tag{5.37}$$

Finally, contracting with $j^\mu = nu^\mu$ and using (5.32) produces the relativistic, non-Abelian Euler equation:

$$\frac{nu^\mu}{c^2} \partial_\mu (u_\nu f'(n)) - n \partial_\nu f'(n) = 2 \operatorname{Tr} J^\mu F_{\mu\nu}. \tag{5.38}$$

The left-hand side is of the form of the usual Abelian Euler equation; the right-hand side describes the non-Abelian Lorentz force acting on the charged fluid.

The curved space generalization of the action for the Lagrangian (5.28) is given by

$$I = \int \sqrt{-\eta} d^4x (j^\mu 2 \operatorname{Tr} Kg^{-1}D_\mu g - f(n)) + I_{\text{gauge}}, \tag{5.39}$$

where $n = \sqrt{j^\mu j^\nu \eta_{\mu\nu}}$. (We use η for the metric even in curved space to avoid confusion with g , the group element.) The variation of this action with respect to the metric, $\delta I = -\frac{1}{2} \int \sqrt{-\eta} \delta \eta_{\mu\nu} \Theta^{\mu\nu}$ identifying the total energy–momentum tensor $\Theta^{\mu\nu}$ as

$$\begin{aligned} \Theta^{\mu\nu} &= \theta^{\mu\nu} + \theta_{\text{gauge}}^{\mu\nu}, \\ \theta^{\mu\nu} &= -\eta^{\mu\nu} [n f'(n) - f(n)] + \frac{u^\mu u^\nu}{c^2} n f'(n). \end{aligned} \tag{5.40}$$

We have used equation (5.34) to eliminate $\operatorname{Tr} Kg^{-1}D_\mu g$ in the matter part of the energy–momentum tensor $\theta^{\mu\nu}$. This behaves as the corresponding Abelian expression (1.101), except that now there is an interaction with a non-Abelian gauge field.

The divergence of $\theta^{\mu\nu}$ entails two independent parts: one proportional to u^ν and the other orthogonal to it, compare (1.108):

$$\partial_\mu \theta^{\mu\nu} = \partial_\mu (nu^\mu) \frac{u^\nu f'(n)}{c^2} + n \left[\frac{u^\mu}{c^2} \partial_\mu (u^\nu f'(n)) - \partial^\nu f'(n) \right]. \tag{5.41}$$

The first vanishes by virtue of (5.33) and the rest is evaluated from the Euler equation (5.38), leaving

$$\partial_\mu \theta^{\mu\nu} = 2 \operatorname{Tr} J_\mu F^{\mu\nu}, \tag{5.42}$$

which is cancelled by the divergence of the gauge-field energy–momentum tensor:

$$\partial_\mu \theta_{\text{gauge}}^{\mu\nu} = -2 \operatorname{Tr} J_\mu F^{\mu\nu}. \tag{5.43}$$

Thus we have conservation of the total energy–momentum tensor.

We record the nonrelativistic limit of the Euler equation (5.38). For small velocities, we may write, as in section 1.4,

$$n \approx \rho - \frac{1}{2c^2} \rho \mathbf{v}^2 \quad u^\mu \approx (c, \mathbf{v}). \quad (5.44)$$

Further, we take $f(n)$ to be of the form

$$f = nc^2 + V(n). \quad (5.45)$$

With these simplifications, we find that the nonrelativistic limit for the spatial component of (5.38) gives the Euler equation with a non-Abelian Lorentz force

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = \text{force} + Q_a \mathbf{E}^a + \frac{\mathbf{v}}{c} \times Q_a \mathbf{B}^a, \quad (5.46)$$

where force is the pressure force coming from the potential V (and is therefore Abelian in nature), while non-Abelian force terms involve the chromoelectric and chromomagnetic fields:

$$E_a^i = cF_{0i}^a, \quad B_a^i = -\frac{1}{2} \epsilon^{ijk} F_{jk}^a. \quad (5.47)$$

It is seen that the non-Abelian fluid moves effectively in a single direction specified by $\mathbf{j} = \rho \mathbf{v}$. Nevertheless, it experiences a non-Abelian Lorentz force.

Even though we wrote down the curved space action (5.39) primarily to obtain the energy–momentum tensor, we note that it may be useful for relativistic astrophysics with the quark–gluon plasma as in early universe or perhaps in the interior of neutron stars.

5.4.2. Multi-component flow. We now return to the more general multi-component action (5.23). The current which couples to the gauge potential is now

$$J^\mu = \sum_{s=1}^r Q_{(s)} j_{(s)}^\mu, \quad \text{with } Q_{(s)} = g K_{(s)} g^{-1}. \quad (5.48)$$

Arbitrary variation of g ensures that (5.48) is covariantly conserved, $D_\mu J^\mu = 0$, but we also need the conservation of individual $j_{(s)}^\mu$. This is achieved by considering special variations of g of the form $\delta_{(s)} g = g K_{(s)} \lambda_{(s)}$. These variations of g lead to

$$\partial_\mu j_{(s)}^\mu = \dot{\rho}_{(s)} + \nabla \cdot (\mathbf{v}_{(s)} \rho_{(s)}) = 0. \quad (5.49)$$

The fluid Wong equation which follows from the conservation of the Abelian and non-Abelian currents now reads

$$\sum_{s=1}^r j_{(s)}^\mu D_\mu Q_{(s)} = 0. \quad (5.50)$$

Varying the individual $j_{(s)}^\mu$ in (5.24) produces the Bernoulli equations

$$2 \text{Tr} Q_{(s)} (D_\mu g) g^{-1} = \frac{u_\mu}{c^2} f^{(s)}, \quad (5.51)$$

where

$$f^{(s)} \equiv \frac{\partial}{\partial n_{(s)}} f(n_{(1)}, n_{(2)}, \dots, n_{(r)}). \quad (5.52)$$

As in the single-channel case, the curl of equation (5.51) can be cast in the form

$$\frac{1}{c^2} \{ \partial^\mu (u_{(s)}^\nu f^{(s)}) - \partial^\nu (u_{(s)}^\mu f^{(s)}) \} = 2 \text{Tr} ((D^\mu Q_{(s)}) (D^\nu g) g^{-1} + Q_{(s)} F^{\mu\nu}). \quad (5.53)$$

When contracted with $j_{(s)}^\mu = n_{(s)} u_{(s)}^\mu$, this leaves

$$\frac{n_{(s)} u_{(s)}^\mu}{c^2} \partial_\mu (u_{(s)}^\nu f^{(s)}) - n_{(s)} \partial^\nu f^{(s)} = j_{\mu(s)} 2 \text{Tr} ((D^\mu Q_{(s)}) (D^\nu g) g^{-1} + Q_{(s)} F^{\mu\nu}). \quad (5.54)$$

However, unlike in the single-channel case, the right-hand side does not simplify since $j_{\mu(s)} Q_{(s)}$ cannot be replaced by J_{μ} because the latter requires summing over s . Also the first right-hand term in (5.54) does not vanish since (5.50) requires summation over s . Equations (5.54) are the Euler equations for the multi-component non-Abelian fluid.

The matter part of the energy–momentum tensor is now given as

$$\theta^{\mu\nu} = -g^{\mu\nu} \left(\sum_{s=1}^r n_{(s)} f^{(s)} - f \right) + \sum_{s=1}^r \frac{u_{(s)}^{\mu} u_{(s)}^{\nu}}{c^2} n_{(s)} f^{(s)}. \tag{5.55}$$

Its divergence is of the form (5.41):

$$\partial_{\mu} \theta^{\mu\nu} = \sum_{s=1}^r \left\{ (\partial_{\mu} (n_{(s)} u_{(s)}^{\mu})) \frac{u_{(s)}^{\nu} f^{(s)}}{c^2} + n_{(s)} \left[\frac{u_{(s)}^{\mu}}{c^2} \partial_{\mu} (u_{(s)}^{\nu} f^{(s)}) - \partial^{\nu} f^{(s)} \right] \right\}. \tag{5.56}$$

Using (5.49) and (5.54), the right-hand side of this equation is evaluated as

$$\sum_{s=1}^r j_{\mu(s)} 2 \operatorname{Tr}((D^{\mu} Q_{(s)})(D^{\nu} g)g^{-1} + Q_{(s)} F^{\mu\nu}).$$

Since now we are summing over all channels, it follows from (5.48) and (5.50) that, as before,

$$\partial_{\mu} \theta^{\mu\nu} = 2 \operatorname{Tr} J_{\mu} F^{\mu\nu}. \tag{5.57}$$

Some simplifications which lead to a more transparent physical picture occur if the dynamical potential separates

$$f(n_{(1)}, \dots, n_{(r)}) = \sum_{s=1}^r f_{(s)}(n_{(s)}) \quad f^{(s)} = f'_{(s)}. \tag{5.58}$$

Then the left-hand side of (5.54) refers only to variables labelled s , while the right-hand side may be rewritten with the help of the generalized Wong equation (5.50) to give

$$\begin{aligned} & \frac{n_{(s)} u_{(s)}^{\mu}}{c^2} \partial_{\mu} (u_{(s)}^{\nu} f'_{(s)}) - n_{(s)} \partial^{\nu} f'_{(s)} \\ &= 2 \operatorname{Tr} \left(J_{\mu} F^{\mu\nu} - \sum_{s' \neq s}^r j_{\mu(s')} (Q_{(s')} F^{\mu\nu} + (D^{\mu} Q_{(s')})(D^{\nu} g)g^{-1}) \right). \end{aligned} \tag{5.59}$$

Thus in addition to the Lorentz force, there are forces arising from the other channels $s' \neq s$.

Note that for separated dynamics (5.58), the energy–momentum tensor also separates,

$$\theta^{\mu\nu} = \sum_{s=1}^r \theta_{(s)}^{\mu\nu} = \sum_{s=1}^r \left\{ -\eta^{\mu\nu} (n_{(s)} f'_{(s)} - f_{(s)}(n_{(s)})) + \frac{u_{(s)}^{\mu} u_{(s)}^{\nu}}{c^2} n_{(s)} f'_{(s)} \right\}, \tag{5.60}$$

but the divergence of each individual $\theta_{(s)}^{\mu\nu}$ does not vanish. This is just as expected since energy can now be exchanged between the different channels and with the gauge field; this is also evident from the equation of motion (5.59). It is clear that this fluid moves with r different velocities $\mathbf{v}_{(s)}$.

The single-channel Euler equation (5.38) is expressed in terms of physically relevant quantities (currents, chromomagnetic fields); the many-channel equation (5.54) involves, additionally, the gauge group element g . One may simplify that equation by going to special gauge, for example $g = I$, so that the right-hand side of (5.54) reduces to

$$j_{(s)}^{\mu} 2 \operatorname{Tr}((D_{\mu} Q_{(s)})(D_{\nu} g)g^{-1} + Q_{(s)} F_{\mu\nu}) = j_{(s)}^{\mu} 2 \operatorname{Tr} K_{(s)} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) \tag{5.61}$$

while the Wong equation (5.50) becomes

$$\sum_{s=1}^r j_{(s)}^\mu [A_\mu, K_{(s)}] = 0. \quad (5.62)$$

It is interesting that in this gauge the non-linear terms in $F_{\mu\nu}$ disappear.

Observe that the inclusion of the α -, β -components of the current, or the use of the Lagrangian (5.24), does not change the form of the equations of motion for fluids when expressed in terms of the velocities and densities. The expressions for these quantities in terms of the group parameters and α , β will, of course, be altered.

6. Noncommutative fluids

Most of the fluids examined so far arise out of underlying particle systems. (The sole exception is the Madelung-like construction based on an underlying field—the Schrödinger field (both Abelian and non-Abelian)—discussed in appendix G.) This becomes manifest in the Lagrange description, in which the coordinates of the underlying particle substratum are explicitly involved. The transition to the Euler description, then, allows us to express the system in terms of purely fluid quantities, namely density and velocity.

It is possible to consider fluids whose Euler description does not derive from an explicit underlying Lagrangian description. Although such fluids would look conventional in terms of density and velocity, they would nevertheless be effective descriptions of exotic underlying theories.

A concrete and interesting realization of this is noncommutative fluids. In these, the fundamental degrees of freedom represent ‘particles’ on a noncommutative space, in which coordinates become noncommuting operators and the notion of points breaks down.

Noncommutative spaces were introduced by Heisenberg to ameliorate ultraviolet infinities in quantum field theory. They have been considered in mathematics and they arise as particular projections of quantum systems in an intense magnetic field (see section 7.2) and also as special limits of string theory. Gauge theory on these spaces becomes particularly attractive, since it fuses spatial and internal degrees of freedom into one coherent formalism. As we shall demonstrate, such theories can be viewed as noncommutative fluids and the transition to the Euler description will emerge as the transformation mapping them to equivalent commutative effective theories, known to the string literature as the Seiberg–Witten map. The presentation follows [44].

6.1. Review of noncommutative spaces

Noncommutative spaces are described in terms of coordinates that are noncommuting operators. In the simplest realization, ‘flat’ noncommuting coordinates are characterized by a constant, antisymmetric tensor θ^{ij} :

$$[x^i, x^j] = i\theta^{ij}. \quad (6.1)$$

We must specify whether θ possesses an inverse ω :

$$\theta^{ij}\omega_{jk} = \delta_k^i. \quad (6.2)$$

An inverse can exist in even dimensions, provided θ is non-singular, but ω will not exist in odd dimensions, where the antisymmetric θ always possesses a zero mode. We shall assume the generic situation: nondegenerate θ with no zero modes in even dimensions, where we give a Euclidean treatment—all coordinates are spatial; while in odd dimensions there is one zero

mode, chosen in the time direction. Therefore, in all dimensions only spatial variables are noncommuting.

Fields f are defined as arbitrary functions of the operator coordinates x^i (and t in odd dimensions). They can be expressed in a Taylor expansion as sums of monomials involving (ordered) products of the coordinates. They are, effectively, themselves operators on the same footing as the x^i (explicitly depending on the commuting coordinate t in odd dimensions). The derivative of a function $\partial_i f$ is defined, as usual, by eliminating x^i once from each place in monomials in which it appears. Equivalently, this can be achieved through the adjoint action of the x^j themselves,

$$\partial_i f = -i\omega_{ij}[x^j, f], \tag{6.3}$$

which clearly has the desired effect on monomials. The volume integral over the full noncommutative space, on the other hand, can be expressed as the trace of the corresponding operator over a Heisenberg-like Hilbert space on which the x^i act:

$$\int f = \sqrt{\det(2\pi\theta)} \text{Tr} f. \tag{6.4}$$

Products of fields in the above space are defined as the usual ordered products of the corresponding operators. We may trade the above noncommutative description for one involving usual, commuting functions by introducing a new, non-local, noncommutative product between functions, called star-product. First, order all monomials in a fully symmetric way in the x^i ('Weyl ordering'), which can always be achieved by using (6.1). This creates a one-to-one correspondence between noncommutative operators and commutative functions. The derivative and volume integral of the noncommutative function as defined in (6.3) and (6.4) map, in fact, to the usual derivative and integral of the commutative function f . The product of two operators f and g , on the other hand, upon Weyl reordering, corresponds to a new function, called the star-product of the corresponding commutative functions f and g and denoted by $f \star g$, and is defined by

$$(f \star g)(x) = \exp\left(\frac{i}{2}\theta^{ij}\right) \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} f(x)g(y)|_{x=y}. \tag{6.5}$$

We may further define the \star -commutator of two functions as their antisymmetrized \star -product:

$$[f, g]_\star = f \star g - g \star f. \tag{6.6}$$

The noncommuting coordinates x^i , in particular, map to the usual commuting coordinates and their \star -commutator reproduces the noncommutative space relations (6.1):

$$[x^i, x^j]_\star = i\theta^{ij}. \tag{6.7}$$

The two formulations, operator and star, are obviously equivalent and will be used interchangeably.

For future use let us record here the formulae of noncommuting, electrodynamics in four-dimensional space. The vector potential \hat{A}_μ is a function of the noncommuting coordinates x^μ , and it undergoes gauge transformations, which infinitesimally read

$$\delta \hat{A}_\mu = \partial_\mu \hat{\lambda} - i[\hat{A}_\mu, \hat{\lambda}]_\star \equiv D_\mu \hat{\lambda}. \tag{6.8}$$

Here $\hat{\lambda}$ is the gauge function, also depending on the noncommuting coordinates. A field strength $\hat{F}_{\mu\nu}$ is defined so that it transforms covariantly: under gauge transformations

$$\delta \hat{F}_{\mu\nu} = -i[\hat{F}_{\mu\nu}, \hat{\lambda}]_\star. \tag{6.9}$$

The formula for $\hat{F}_{\mu\nu}$ therefore reads

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_\star, \tag{6.10}$$

and the equations satisfied by the $\hat{F}_{\mu\nu}$ are

$$\frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}D_\beta\hat{F}_{\mu\nu} = 0, \quad (6.11)$$

$$D_\mu\hat{F}^{\mu\nu} = \hat{J}^\nu, \quad (6.12)$$

where \hat{J}^ν is a noncommuting source current. Note that even though we are dealing with electrodynamics, the noncommutativity requires a non-Abelian, Yang–Mills-like structure for the relevant expressions.

The noncommutative space structure (6.1) remains invariant under a group of transformations. Subjecting the spatial coordinates to an infinitesimal coordinate transformation

$$\delta\mathbf{x} = -\mathbf{f}(\mathbf{x}), \quad (6.13)$$

for some operator functions f^i , and requiring that (6.1) remain unchanged results in the condition

$$-[f^i(\mathbf{x}), x^j] - [x^i, f^j(\mathbf{x})] = 0, \quad (6.14)$$

which in turn implies by (6.1) that

$$-\partial_k f^i(\mathbf{x})\theta^{kj} - \partial_k f^j(\mathbf{x})\theta^{ik} = 0. \quad (6.15)$$

The left-hand side is recognized as the Lie derivative of a contravariant tensor,

$$L_f\theta^{ij} = f^k\partial_k\theta^{ij} - \partial_k f^i\theta^{kj} - \partial_k f^j\theta^{ik}, \quad (6.16)$$

with the first term on the right-hand side vanishing since θ is constant. So the noncommutative algebra (6.1) is preserved by those coordinate transformations that leave θ invariant: $L_f\theta = 0$.

To solve for \mathbf{f} in even dimensions, we define

$$f^i = \theta^{ij}g_j \quad (i, j = 1, \dots, 2n). \quad (6.17)$$

This entails no loss of generality, because θ is non-singular (by hypothesis). Then (6.15) becomes

$$\theta^{i\ell}\partial_k g_\ell\theta^{kj} + \theta^{j\ell}\partial_k g_\ell\theta^{ik} = 0. \quad (6.18a)$$

Because θ is non-singular and antisymmetric, this implies

$$\partial_k g_\ell - \partial_\ell g_k = 0, \quad (6.18b)$$

or

$$g_\ell = \partial_\ell\phi. \quad (6.19)$$

Thus we have

$$f^i = \theta^{ij}\partial_j\phi \quad (6.20)$$

for the ct s (in even dimensions) that leave θ invariant. Since

$$\nabla \cdot \mathbf{f} = 0 \quad (6.21)$$

the transformations are volume preserving; the Jacobian of the finite diffeomorphism is unity. However, except in two dimensions, these are not the most general volume-preserving transformations. Nevertheless, they form a group: the Lie bracket of two transformations like (6.15), $f_1^i = \theta^{ij}\partial_j\phi_1$ and $f_2^i = \theta^{ij}\partial_j\phi_2$, takes the same form, $\theta^{ij}\partial_j(\theta^{k\ell}\partial_k\phi_1\partial_\ell\phi_2)$. The group is the symplectic subgroup of volume-preserving diffeomorphisms that also preserve θ^{ij} . In two dimensions, where we can set $\theta^{ij} = \theta\varepsilon^{ij}$, the above transformations exhaust all the area-preserving transformations.

In odd dimensions, where (by assumption) θ possesses a single zero mode, for definiteness we orient the coordinates so that the zero mode lies in the first direction (labelled $0 \rightarrow$ time) and θ , confined to the remaining (spatial) dimensions, is non-singular:

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \theta^{ij} \end{pmatrix} \quad (i, j = 1, \dots, 2n) \quad \theta^{ij} \omega_{jk} = \delta_k^i. \quad (6.22)$$

The infinitesimal diffeomorphisms that preserve θ are

$$f^\mu = \begin{cases} f(t) \\ \theta^{ij} \frac{\partial}{\partial x^j} \phi(t, \mathbf{x}). \end{cases} \quad (6.23)$$

These still form a group. Two transformations, (f_1, ϕ_1) and (f_2, ϕ_2) , possess a Lie bracket of the same form (6.23), with $(f_2 \partial_t f_1 - f_1 \partial_t f_2, f_2 \partial_t \phi_1 - f_1 \partial_t \phi_2 + \theta^{k\ell} \partial_k \phi_1 \partial_\ell \phi_2)$. But the spacetime volume is not preserved: $\partial_\mu f^\mu \neq 0$. (Of course, at fixed time, the spatial volume is preserved.)

Unit-Jacobian diffeomorphisms also leave invariant the equations for an ideal fluid, in the Lagrange formulation of fluid mechanics, and in particular a planar (two-dimensional) fluid supports area-preserving diffeomorphisms; see section 1.2.2. This coincidence of invariance suggests that other aspects of noncommutativity possess analogues in the theory of fluids, whose familiar features can therefore clarify some obscurities of noncommutativity. (A similar point of view concerning the quantum Hall effect was taken in [45].) We shall explore this connection and shall demonstrate that the Seiberg–Witten map [46] between noncommuting and commuting gauge fields corresponds to the mapping between the Lagrange and Euler formulations of fluid mechanics. We shall obtain a simple derivation of the explicit ‘solution’ to the Seiberg–Witten map in even dimensions [47] and will extend it to odd dimensions.

The two formulations of fluid dynamics (Lagrange and Euler) can be put in the proper context for the noncommutative space setting. The natural Poisson (commutator) structure, present in the Lagrange description of a fluid, and the possibility of introducing a vector potential to describe the evolution of comoving coordinates, will be recognized as classical precursors of analogous noncommuting entities. Within this framework, we shall show how noncommuting gauge fields respond to coordinate transformations, generalizing previously established results [48].

As explained in section 1.1, the Lagrange description uses the coordinates of the particles comprising the fluid: $\mathbf{X}(t, \mathbf{x})$. These are labelled by the comoving coordinates \mathbf{x} , which are the coordinates of some initial reference configuration, e.g., $\mathbf{X}(0, \mathbf{x}) = \mathbf{x}$, see (1.14). We may parametrize the evolution of \mathbf{X} by defining

$$X^i(t, \mathbf{x}) = x^i + \theta^{ij} \hat{A}_j(t, \mathbf{x}) \quad (6.24)$$

which loses no generality provided θ is non-singular. As will be seen below, $\hat{\mathbf{A}}$ behaves as a noncommuting, Abelian vector potential.

6.2. Noncommutative gauge theory

6.2.1. Commuting theory with Poisson structure. We introduce into the Lagrange fluid description the (non-singular) antisymmetric tensor θ . This allows for a natural definition of a Poisson bracket, which may be viewed as a classical precursor of the noncommutativity of coordinates. We define the bracket by

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \theta^{ij} \frac{\partial \mathcal{O}_1}{\partial x^i} \frac{\partial \mathcal{O}_2}{\partial x^j} \quad (6.25)$$

so that

$$\{x^i, x^j\} = \theta^{ij}. \quad (6.26)$$

It follows from the definition (6.24) that

$$\{X^i, X^j\} = \theta^{ij} + \theta^{ik}\theta^{j\ell}\hat{F}_{k\ell} \quad (6.27)$$

with

$$\hat{F}_{ij} = \frac{\partial}{\partial x^i}\hat{A}_j - \frac{\partial}{\partial x^j}\hat{A}_i + \{\hat{A}_i, \hat{A}_j\}. \quad (6.28)$$

It is seen that the structure of the gauge field \hat{F} is as in a noncommuting theory, with the Poisson bracket replacing the $*$ -commutator of two potentials \hat{A} , compare (6.10). Also, in the limit that the deviation of \mathbf{X} from the reference configuration \mathbf{x} is small, that is, for small \hat{A} , we recover a conventional Abelian gauge field.

The above formulae are understood to hold either in even dimensions for a purely spatial Euclidean formulation (there is no time variable) or in odd-dimensional spacetime for spatial components. (\mathbf{X} and \mathbf{x} are spatial vectors, without time components.)

6.2.2. *Coordinate transformations in the commuting theory (even dimensions).* In even dimensions, the θ -preserving transverse diffeomorphism, which also implements the reparametrization symmetry of the Lagrange fluid, acts on \mathbf{X} according to (1.27) and (6.20) as

$$\delta_\theta \mathbf{X} = \mathbf{f} \cdot \nabla \mathbf{X} = \theta^{ij} \frac{\partial \mathbf{X}}{\partial x^i} \frac{\partial \phi}{\partial x^j}. \quad (6.29)$$

This may be presented with the help of the bracket defined in (6.25).

$$\delta_\phi \mathbf{X}(\mathbf{x}) = \theta^{ij} \frac{\partial \mathbf{X}(\mathbf{x})}{\partial x^i} \frac{\partial \phi(\mathbf{x})}{\partial x^j} = \{\mathbf{X}(\mathbf{x}), \phi(\mathbf{x})\}. \quad (6.30)$$

Because $\delta \mathbf{X}$ compares the transformed and untransformed \mathbf{X} at the same argument, $\delta \hat{A}_i = \omega_{ij} \delta X^j$ and the volume-preserving diffeomorphism (6.29), (6.30) induces a gauge transformation on \hat{A} : (compare (6.9), (6.10)):

$$\delta_\phi \hat{A}(\mathbf{x}) = \nabla \phi(\mathbf{x}) + \{\hat{A}(\mathbf{x}), \phi(\mathbf{x})\} \equiv \mathbf{D}\phi \quad (6.31a)$$

$$\delta_\phi \hat{F}_{ij}(\mathbf{x}) = \{\hat{F}_{ij}(\mathbf{x}), \phi(\mathbf{x})\}. \quad (6.31b)$$

We see that the dynamically sterile relabelling diffeomorphism of the parameters in the Lagrange fluid leads to an equally sterile gauge transformation, under which \mathbf{X} and \hat{F} transform covariantly, as in (6.30), (6.31b).

Next we consider a diffeomorphism of the target space:

$$\delta_f \mathbf{X} = -\mathbf{f}(\mathbf{X}). \quad (6.32)$$

In contrast to the previous relabellings, this transformation is dynamical, deforming the fluid configuration. Quantities

$$C_n(\mathbf{X}) = \frac{1}{2^n n!} \varepsilon_{i_1 j_1 \dots i_n j_n} \{X^{i_1}, X^{j_1}\} \dots \{X^{i_n}, X^{j_n}\}, \quad (6.33)$$

which are defined in $d = 2n$ dimensions, respond to the transformation (6.32) in a noteworthy fashion. One verifies that

$$\delta_f C_n(\mathbf{X}) = -\nabla \cdot \mathbf{f}(\mathbf{X}) C_n(\mathbf{X}), \quad (6.34)$$

so that transverse (volume-preserving) target-space diffeomorphisms leave C_n invariant. Equation (6.34) is most easily established by recognizing that

$$C_n(\mathbf{X}) = \text{Pfaff}\{X^i, X^j\} = \det^{1/2}\{X^i, X^j\} = \det^{1/2}\theta \det \frac{\partial X^i}{\partial x^j}. \quad (6.35)$$

The significance of these transformations is evident from (1.22), which shows that $1/\rho(\mathbf{r}) = C_n(\mathbf{X})|_{\mathbf{x}=\chi(\mathbf{r})}$ when $\det^{1/2}\theta$ is identified with $1/\rho_0$. The transformation law for ρ under transverse target-space diffeomorphisms becomes

$$\delta_{\mathbf{f}}\rho(\mathbf{r}) = \mathbf{f} \cdot \nabla\rho(\mathbf{r}). \quad (6.36)$$

It follows that this transformation leaves invariant all terms in the Lagrangian that depend only on ρ (like the potential in (1.36)).

When we restrict the transverse, target-space diffeomorphisms to those that also leave θ invariant, i.e., (6.20) (of course in two dimensions this is not a restriction), further quantities are left invariant. These are constructed as in (6.33), but with any number of brackets $\{X^i, X^j\}$ replaced by θ^{ij} .

It is interesting to combine the diffeomorphism of the parameter space with that of the target space, for a simultaneous transformation on both spaces. To this end we chose the form of the target-space transformation to coincide with that of the reparametrization/relabelling transformation:

$$\mathbf{f}^i(\mathbf{X}) = \theta^{ij} \frac{\partial\phi(\mathbf{X})}{\partial X^j}. \quad (6.37)$$

As we shall show below, this results in a gauge-covariant coordinate transformation on the vector potential $\hat{\mathbf{A}}$, once a further gauge transformation is carried out. Thus we consider $\Delta \equiv \delta_\phi + \delta_{\mathbf{f}}$,

$$\Delta X^i = \{X^i, \phi(\mathbf{x})\} - \theta^{ij} \frac{\partial\phi(\mathbf{X})}{\partial X^j}. \quad (6.38)$$

(Note that any deviation of $\mathbf{f}^i(\mathbf{X})$ from $\theta^{ij}\partial\phi(\mathbf{X})/\partial X^j$ may be attributed to ϕ , and can be removed by a further gauge transformation.) However, covariance is not preserved in (6.38): \mathbf{X} on the left-hand side is covariant, but on the right-hand side in the Poisson bracket there occurs $\phi(\mathbf{x})$, which is not covariant. The defect may be remedied by combining $\Delta\mathbf{X}$ with a further gauge transformation:

$$\delta_{\text{gauge}}\mathbf{X} = \{\mathbf{X}, \phi(\mathbf{X}) - \phi(\mathbf{x})\}, \quad (6.39)$$

so that in $\Delta + \delta_{\text{gauge}} \equiv \hat{\delta}$ we have a covariant transformation rule,

$$\hat{\delta}X^i = \{X^i, \phi(\mathbf{X})\} - \theta^{ij} \frac{\partial\phi(\mathbf{X})}{\partial X^j}, \quad (6.40)$$

which in turn implies that $\hat{\mathbf{A}}$ transforms as

$$\hat{\delta}\hat{A}_i = \omega_{ij}\{X^j, \phi(\mathbf{X})\} - \frac{\partial\phi(\mathbf{X})}{\partial X^i}. \quad (6.41)$$

To recognize this transformation more clearly, we present it as

$$\hat{\delta}\hat{A}_i = \omega_{ij}\{X^j, X^k\} \frac{\partial\phi(\mathbf{X})}{\partial X^k} - \frac{\partial\phi(\mathbf{X})}{\partial X^i}, \quad (6.42a)$$

and use (6.27) to find

$$\hat{\delta}\hat{A}_i = \theta^{k\ell} \hat{F}_{i\ell} \frac{\partial\phi(\mathbf{X})}{\partial X^k} = f^k(\mathbf{X}) \hat{F}_{ki}. \quad (6.42b)$$

Note that in the final expression (6.42b) the response of $\hat{\mathbf{A}}$ is entirely covariant: it involves the covariant curvature \hat{F} and the diffeomorphism function \mathbf{f} evaluated on the covariant argument \mathbf{X} coordinate transformation. This expression records precisely a gauge-covariant transformation. (This was derived by a somewhat different method [48].)

6.2.3. *Coordinate transformations in the noncommuting theory with *-products (even dimensions).* The above development may be taken over directly into a noncommutative field theory by replacing Poisson brackets by $-i$ times $*$ -commutators, so that (6.26) goes over into (6.1). Equation (6.24) remains and (6.27), (6.28) become

$$[X^i, X^j]_* = i\theta^{ij} + i\theta^{ik}\theta^{j\ell}\hat{F}_{k\ell} \quad (6.43)$$

with \hat{F} given by (6.10). The covariant transformation rules (6.40) and (6.41) may be used in the noncommutative context, provided a sensible ordering prescription is set for $\phi(\mathbf{X})$. This we do as follows. Define

$$\Phi = \int dx \phi(\mathbf{X}) \quad (6.44a)$$

where $\phi(\mathbf{X})$ is a series of (star) powers of \mathbf{X} :

$$\phi(\mathbf{X}) = c + c_i X^i + \frac{1}{2}c_{ij} X^i \star X^j + \frac{1}{3}c_{ijk} X^i \star X^j \star X^k + \dots \quad (6.44b)$$

(We are not concerned about convergence of the integral (6.44a), since we are interested in local quantities like (6.44b) or (6.47) below.) The integration over \mathbf{x} (the argument of \mathbf{X}) ensures that Φ is invariant (in an operator formalism the integral becomes the trace of the operators). The c -coefficients in (6.44b) are required to be invariant against cyclic index shuffling (so that Φ and ϕ possess the same number of free parameters). Also we require ϕ to be Hermitian. (This ensures, e.g., that c_{ij} is real symmetric; that $\text{Re } c_{ijk}$ is entirely symmetric and that $\text{Im } c_{ijk}$ is entirely antisymmetric (which is impossible in two dimensions).) Then (6.40) and (6.41) become

$$\hat{\delta}X^i = -i[X^i, \phi(\mathbf{X})]_* - \theta^{ij} \frac{\delta\Phi}{\delta X^j} \quad (6.45)$$

$$\hat{\delta}\hat{A}_i = -i\omega_{ij}[X^j, \phi(\mathbf{X})]_* - \frac{\delta\Phi}{\delta X^i} \quad (6.46)$$

where now the last entries employ a functional derivative:

$$\frac{\delta\Phi}{\delta X^i} = c_i + c_{ij}X^j + c_{ijk}X^j \star X^k + \dots \quad (6.47)$$

In two dimensions, the ordering prescription (6.44) and its consequence (6.47) preserve the invariance of the $[X^i, X^j]_*$ -commutator against the target-space diffeomorphism (last term in (6.45)). Thereby a property of the classical Poisson bracket (cf (6.34) at $n = 1$) is maintained in the noncommuting theory.

With $\phi(\mathbf{X})$ at most quadratic in \mathbf{X} (\mathbf{f} at most linear), one readily verifies the result in [48].

$$\delta\hat{A}_i = \frac{1}{2}\{f^j(\mathbf{X}) \star \hat{F}_{ji} + \hat{F}_{ji} \star f^j(\mathbf{X})\}. \quad (6.48)$$

But with more general ϕ (\mathbf{f} containing quadratic and higher powers) there arise further reordering terms.

6.2.4. *Coordinate transformations in commuting and noncommuting theories (odd dimensions).* In odd dimensions, with the θ -preserving transformation function given by (6.23), the relabelling transformation on the base space is

$$\begin{aligned} \delta_\phi \mathbf{X}(t, \mathbf{x}) &= \theta^{ij} \frac{\partial}{\partial x^j} \phi(t, \mathbf{x}) \frac{\partial}{\partial x^i} \mathbf{X}(t, \mathbf{x}) + f(t) \dot{\mathbf{X}}(t, \mathbf{x}) \\ &= \{\mathbf{X}(t, \mathbf{x}), \phi(t, \mathbf{x})\} + f(t) \dot{\mathbf{X}}(t, \mathbf{x}). \end{aligned} \quad (6.49)$$

The fluid coordinate \mathbf{X} has components only in the spatial directions. Here the Poisson bracket is defined with the non-singular θ^{ij} .

For the target-space diffeomorphism we again take the formula (6.37), so that the combined, noncovariant transformation $\Delta \equiv \delta_\phi + \delta_f$ reads

$$\Delta X^i = \{X^i, \phi(t, \mathbf{x})\} + f(t)\dot{X}^i(t, \mathbf{x}) - \theta^{ij} \frac{\partial \phi(t, \mathbf{X})}{\partial X^j}. \quad (6.50)$$

This is modified by the gauge transformation

$$\delta_{\text{gauge}} \mathbf{X} = \{\mathbf{X}, \phi(t, \mathbf{X}) - \phi(t, \mathbf{x})\} - \{\mathbf{X}, f(t)\hat{A}_0(t, \mathbf{x})\} \quad (6.51)$$

resulting in the covariant transformation $\Delta + \delta_{\text{gauge}} \equiv \hat{\delta}$:

$$\hat{\delta} X^i = \{X^i, \phi(t, \mathbf{X})\} - \theta^{ij} \frac{\partial \phi(t, \mathbf{X})}{\partial X^j} + f(t)DX^i. \quad (6.52)$$

Here $DX^i = \dot{X}^i + \{\hat{A}_0, X^i\}$, where \hat{A}_0 is a connection introduced to render the time derivative covariant against time-dependent gauge transformations, generated by ϕ . This is achieved when the gauge transformation law for \hat{A}_0 is

$$\delta_\phi \hat{A}_0 = \dot{\phi} + \{\hat{A}_0, \phi\}. \quad (6.53)$$

The spatial components of the vector potential are introduced as before in (6.24), and DX^i becomes

$$DX^i = \theta^{ij}(\dot{\hat{A}}_j - \partial_j \hat{A}_0 + \{\hat{A}_0, \hat{A}_j\}) = \theta^{ij} \hat{F}_{0j}. \quad (6.54)$$

The covariant transformation law of $\hat{\mathbf{A}}$ follows from (6.43), (6.52) and (6.54):

$$\begin{aligned} \hat{\delta} \hat{A}_i &= \omega_{ij} \{X^j, \phi(t, \mathbf{X})\} - \frac{\partial \phi(t, \mathbf{X})}{\partial X^i} + \omega_{ij} f(t)DX^j \\ &= f^j(t, \mathbf{X})\hat{F}_{ji} + f(t)\hat{F}_{0i} = f^\mu(t, \mathbf{X})\hat{F}_{\mu i}. \end{aligned} \quad (6.55)$$

It remains to fix the transformation law of \hat{A}_0 . This requires specifying $\delta_f \hat{A}_0$. Since

$$\delta_f \hat{A}_i = -\frac{\partial \phi(t, \mathbf{X})}{\partial X^i} \quad (6.56)$$

it is natural to take

$$\delta_f \hat{A}_0 = -\dot{\phi}(t, \mathbf{X}) \quad (6.57)$$

(The time derivative acts on the first argument only.) Thus we have from (6.53) and (6.57)

$$\Delta \hat{A}_0 = \dot{\phi}(t, \mathbf{x}) + \{\hat{A}_0, \phi(t, \mathbf{x})\} - \dot{\phi}(t, \mathbf{X}). \quad (6.58)$$

After adding to this a gauge transformation generated by $\phi(t, \mathbf{X}) - \phi(t, \mathbf{x})$ we are left with

$$\begin{aligned} \hat{\delta} \hat{A}_0 &= \frac{\partial \phi(t, \mathbf{X})}{\partial X^i} \dot{X}^i + \{\hat{A}_0, \phi(t, \mathbf{X})\} \\ &= \frac{\partial \phi}{\partial X^i} DX^i = f^i(t, \mathbf{X})\hat{F}_{i0} = f^\mu(t, \mathbf{X})\hat{F}_{\mu 0}. \end{aligned} \quad (6.59)$$

Equations (6.55) and (6.59) coincide with the formula obtained in a conventional commuting gauge theory [49].

Similar results follow within the noncommuting formalism, once the now familiar ordering prescription is given for $\phi(t, \mathbf{X})$ and $\phi = \int dx \phi(t, \mathbf{X})$. In the noncommutative theory (6.55) and (6.59) are regained, up to reordering terms.

6.3. Seiberg–Witten map = Euler fluid–Lagrange fluid map

By considering various limits within string theory, Seiberg and Witten found that noncommuting fields can be mapped onto non-local functions of commuting fields [46]. The principle behind the mapping can be stated as a requirement of stability against gauge transformations in the following sense. Consider the noncommuting gauge potential \hat{A}_μ to be a functional of the commuting gauge potential A_μ and of θ . It is then required that a commuting gauge transformation performed on the commuting gauge potential with commuting gauge function $\lambda : A_\mu \rightarrow A_\mu^\lambda \equiv A_\mu + \partial_\mu \lambda$, can be equivalently achieved by performing noncommuting gauge transformation on the noncommuting gauge potential with noncommuting gauge function $\hat{\lambda} : \hat{A}_\mu \rightarrow \hat{A}_\mu^\lambda \equiv (e^{i\hat{\lambda}}) * [\hat{A}_\mu + i\partial_\mu] * (e^{i\hat{\lambda}})^{-1}$. Thus

$$\hat{A}_\mu^{\hat{\lambda}}(A_\mu) = \hat{A}_\mu(A_\mu + \partial_\mu \lambda). \quad (6.60a)$$

This in turn implies a differential equation in θ for \hat{A}_μ ,

$$\frac{\partial \hat{A}_\mu}{\partial \theta^{\alpha\beta}} = -\frac{1}{8} \{ \hat{A}_\alpha, \partial_\beta \hat{A}_\mu + \hat{F}_{\beta\mu} \}_*^+ - (\alpha \leftarrow \beta) \quad (6.60b)$$

with ‘initial’ condition $\hat{A}_\mu|_{\theta=0} = A_\mu$. $\{, \}_*^+$ denotes the $*$ -anticommutator. One can solve (6.60b) order-by-order in θ , with the lowest order solution being

$$\hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) + \dots, \quad (6.61)$$

but in general, the equations for different α and β contained in (6.60b) are not integrable, except in two dimensions where $\theta^{\alpha\beta}$ involves a single quantity $\theta^{\alpha\beta} = \varepsilon^{\alpha\beta} \theta$.

Apart from its intrinsic mathematical interest as providing a connection between commuting and noncommuting fields, the Seiberg–Witten map also serves a practical purpose. The noncommuting field strength is not gauge invariant, rather, just as in Yang–Mills theory, it is gauge covariant: $\hat{F}_{\mu\nu} \rightarrow (e^{i\hat{\lambda}}) * \hat{F}_{\mu\nu} * (e^{i\hat{\lambda}})^{-1}$. But unlike in Yang–Mills theory, there are no local gauge-invariant quantities. To obtain a gauge-invariant result one must integrate over x^μ , or (equivalently) consider the trace over the Hilbert space on which the operators x^μ act. However, if one wishes to compare predictions of noncommuting electrodynamics with those of ordinary commuting electrodynamics (to set limits on the amount of noncommutativity in Nature) non-local quantities are not useful, because the physical content of ordinary electromagnetism is expressed by local quantities (waves, energy and momentum densities, etc). Here the Seiberg–Witten map provides a resolution: map the noncommuting theory onto a commuting one, from which local gauge-invariant quantities may be extracted [50]¹³.

We now show that the inverse Seiberg–Witten map is equivalent to the map between Lagrange and Euler descriptions for fluids. The argument is first presented in two Euclidean dimensions and (2+1)-dimensional spacetime, where (6.60b) is integrable (because it involves a single derivative variable). Then the argument is taken to higher dimensions.

6.3.1. Seiberg–Witten map in (2) and (2 + 1) dimensions. To construct the Seiberg–Witten map in two Euclidean dimensions, we (temporarily) introduce a time dependence in the fluid variables (but not into the diffeomorphism functions—only spatial variables are transformed) and observe that $(c\rho, \mathbf{v}\rho)$ form a conserved 3-vector j^α (also true in the noncommuting theory when an ordered definition for $\delta(\mathbf{X}(t, \mathbf{x}) - \mathbf{r})$ is given—this will be provided below). Therefore, the dual of j^α , $\varepsilon_{\mu\nu\alpha} j^\alpha$, satisfies a Bianchi identity and can be presented as the curl of a potential, apart from additive and multiplicative constants:

¹³ This approach has been taken to study wave propagation in noncommuting Maxwell theory.

$$\varepsilon_{\mu\nu\alpha} j^\alpha \propto F_{\mu\nu} + \text{constant} \tag{6.62}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.63}$$

Note $j^\alpha, F_{\mu\nu}, A_\mu$ are ordinary functions, even in the noncommuting setting, since the noncommuting variables \mathbf{X} are integrands (in an operator formalism, their trace is involved). In particular, the spatial tensor is determined by ρ :

$$\frac{\partial}{\partial r^i} A_j(\mathbf{r}) - \frac{\partial}{\partial r^j} A_i(\mathbf{r}) = F_{ij}(\mathbf{r}) = -\varepsilon_{ij}(\rho - \rho_0) = -\varepsilon_{ij}\rho_0 \left(\int dx \delta(\mathbf{X}(\mathbf{x}) - \mathbf{r}) - 1 \right). \tag{6.64}$$

(The time dependence is now suppressed.) \mathbf{X} contains $\hat{\mathbf{A}}$, as in (6.24). Since \mathbf{X} is (noncommuting) gauge covariant, the integral in (6.64) is (noncommuting) gauge invariant. Therefore, (6.64) serves to define an (inverse) Seiberg–Witten map between the noncommuting (hatted) and commuting (unhatted) variables. The additive ($\varepsilon_{ij}\rho_0$) and multiplicative (-1) constants are fixed by requiring agreement at small $\hat{\mathbf{A}}$. It still remains to give a proper ordering to the δ -function containing \mathbf{X} . This we do by a Fourier transform prescription

$$\int dr e^{i\mathbf{k}\cdot\mathbf{r}} F_{ij}(\mathbf{r}) = -\varepsilon_{ij}\rho_0 \int dx (e^{i\mathbf{k}\cdot\mathbf{X}(x)} - e^{i\mathbf{k}\cdot\mathbf{x}}), \tag{6.65}$$

and the ordering (Weyl ordering) is defined by the expansion of the exponential in (\star -product) powers: $e^{i\mathbf{k}\cdot\mathbf{X}} \equiv 1 + i\mathbf{k}\cdot\mathbf{X} - \frac{1}{2}\mathbf{k}\cdot\mathbf{X}\star\mathbf{k}\cdot\mathbf{X} + \dots$.

When the exponential $e^{i\mathbf{k}\cdot\mathbf{X}}$ is written explicitly in terms of $\hat{\mathbf{A}}$: $\exp_\star i(k_i x^i + \theta k_i \varepsilon^{ij} \hat{A}_j)$, factoring the exponential into $e^{i\mathbf{k}\cdot\mathbf{x}}$ times another factor involves the Baker–Hausdorff lemma, (because the individual terms in the exponent do not \star -commute). This leads to an open Wilson line integral [51]. In that form (6.65) is seen to coincide with the known solution to the Seiberg–Witten map [47], which is now also recognized as nothing but an instance of the Lagrange \rightarrow Euler map of fluid mechanics.

To construct the Seiberg–Witten map in (2+1)-dimensional spacetime we consider the conserved current, defined in (1.16) and (1.17), except that now the time dependence is retained throughout and the derivative is gauged with \hat{A}_0 :

$$\mathbf{j}(t, \mathbf{r}) = \int dx (\dot{\mathbf{X}} + \{\hat{A}_0, \mathbf{X}\}) \delta(\mathbf{X} - \mathbf{r}). \tag{6.66}$$

The operator ordering is prescribed in momentum space with the exponential (Weyl) ordering and (6.66) in the noncommuting theory becomes

$$\mathbf{j}(t, \mathbf{k}) \equiv \int dr e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{j}(t, \mathbf{r}) = \int dx e^{i\mathbf{k}\cdot\mathbf{X}} (\dot{\mathbf{X}} - i[\hat{A}_0, \mathbf{X}]_\star). \tag{6.67}$$

Note that the commutator does not contribute to current conservation because it is separately transverse:

$$\int dx e^{i\mathbf{k}\cdot\mathbf{X}} [\hat{A}_0, \mathbf{k}\cdot\mathbf{X}]_\star = \int dx \hat{A}_0 [\mathbf{k}\cdot\mathbf{X}, e^{i\mathbf{k}\cdot\mathbf{X}}]_\star = 0. \tag{6.68}$$

Therefore the 3-current is conserved as before. Its dual, $\varepsilon_{\mu\nu\alpha} j^\alpha$, satisfies the Bianchi identity, so the Seiberg–Witten mapping reads

$$\begin{aligned} \int dr e^{i\mathbf{k}\cdot\mathbf{r}} \left(1 - \frac{1}{2} \theta^{ij} F_{ij} \right) &= \int dx e^{i\mathbf{k}\cdot\mathbf{X}}, \\ \int dr e^{i\mathbf{k}\cdot\mathbf{r}} F_{0i} &= \omega_{ij} \int dx e^{i\mathbf{k}\cdot\mathbf{X}} (\dot{X}^j - i[\hat{A}_0, X^j]_\star) \\ &= \int dx e^{i\mathbf{k}\cdot\mathbf{X}} \hat{F}_{0i}. \end{aligned} \tag{6.69}$$

Formulae (6.65) and (6.69) may be verified by comparison with the explicit $\mathcal{O}(\theta)$ Seiberg–Witten map (6.61), which for field strengths implies

$$F_{\mu\nu} = \hat{F}_{\mu\nu} - \theta^{\alpha\beta} (\hat{F}_{\alpha\mu} \hat{F}_{\beta\nu} - \hat{A}_\alpha \partial_\beta \hat{F}_{\mu\nu}). \quad (6.70)$$

Upon setting $\theta^{\alpha 0} = 0$, $\theta^{ij} = \theta \varepsilon^{ij}$ and

$$e_{\star}^{ik_i(x^i + \theta^{ij} \hat{A}_j)} = e^{i\mathbf{k}\cdot\mathbf{x}} \left(1 + i\theta k_i \varepsilon^{ij} \hat{A}_j - \frac{1}{2} \theta^2 k_i k_m \varepsilon^{ij} \varepsilon^{mn} \hat{A}_j \hat{A}_n \right) \quad (6.71)$$

it is recognized that (6.65) and (6.69) reproduce (6.70).

6.3.2. Seiberg–Witten map in higher even dimensions. In dimensions higher than three the correspondence between the Bianchi identity and the conservation of particle current is lost. The derivation of the Seiberg–Witten map calls for higher conserved currents, whose duals are 2-forms.

The introduction of such currents can be motivated by starting again from the commutative particle density ρ as expressed in (1.16) and its inverse ρ^{-1} as expressed in (1.22). Their product

$$1 = \int d\mathbf{x} \delta(\mathbf{X} - \mathbf{r}) \det \frac{\partial X^i(\mathbf{x})}{\partial x^j} \quad (6.72)$$

is independent of the fluid profile $\mathbf{X}(\mathbf{x})$ and constitutes a topological invariant. The Jacobian determinant in the above can be expressed in terms of the square-root determinant (Pfaffian) of the antisymmetric matrix $\{X^j, X^k\}$,

$$1 = \frac{\rho_0}{2^n n!} \int d\mathbf{x} \delta(\mathbf{X} - \mathbf{r}) \epsilon_{i_1, j_1, \dots, i_n, j_n} \{X^{i_1}, X^{j_1}\} \dots \{X^{i_n}, X^{j_n}\} = \rho_0 \int d\mathbf{x} \delta(\mathbf{X} - \mathbf{r}) C_n(\mathbf{X}) \quad (6.73)$$

where, in analogy with the two-dimensional case, we identified $\text{Pfaff}(\boldsymbol{\theta})$ with $1/\rho_0$. Removing all n Poisson brackets from the above recovers the full density ρ . The removal of a single Poisson bracket $\{X^i, X^j\}$, then, produces a sort of residual density ρ_{ij} in the corresponding dimensions, which becomes a candidate for the Seiberg–Witten commutative field strength:

$$\rho_{ij} = \frac{\rho_0}{2^{n-1} (n-1)!} \int d\mathbf{x} \delta(\mathbf{X} - \mathbf{r}) \epsilon_{i, j, i_2, j_2, \dots, i_n, j_n} \{X^{i_2}, X^{j_2}\} \dots \{X^{i_n}, X^{j_n}\}. \quad (6.74)$$

The current dual to ρ_{ij} , in momentum space,

$$J^{j_1 \dots j_{2n-2}} = \frac{\rho_0}{2^{n-1} (n-1)!} \int d\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{X}} \{X^{j_1}, X^{j_2}\} \dots \{X^{j_{2n-3}}, X^{j_{2n-2}}\} \quad (6.75)$$

(the indices are fully antisymmetrized) is gauge invariant and conserved, ensuring that ρ_{ij} satisfies the Bianchi identity.

The corresponding current in the noncommutative case can be written by turning products into \star -products and Poisson brackets into $(-i)$ times \star -commutators. The ordering of the exponential and other factors above has to be fixed in a way which ensures that the obtained current is conserved. Various such orderings are possible. For definiteness, we pick the ordering corresponding to the choice made in [47]

$$J^{j_1 \dots j_{2n-2}} = \frac{\rho_0}{(2i)^{n-1}} \int d\mathbf{x} \int_0^1 ds_1 \dots \int_0^1 ds_{n-1} \delta \left(1 - \sum_{i=1}^{n-1} s_i \right) \times e^{is_1 \mathbf{k}\cdot\mathbf{X}} [X^{j_1}, X^{j_2}]_{\star} \dots \star e^{is_{n-1} \mathbf{k}\cdot\mathbf{X}} \star [X^{j_{2n-3}}, X^{j_{2n-2}}]_{\star}. \quad (6.76)$$

This corresponds to Weyl-ordering the exponential and distributing it in all possible ways between the different commutators. Note that the volume of the s_i -integration space reproduces the factor $1/(n-1)!$ present in (6.75).

To express compactly the above and to facilitate the upcoming derivations, we introduce antisymmetric tensor notation. We define the basis 1-tensors v_j representing the derivative vector field ∂_j , and corresponding 1-forms dx^j . We consider the fundamental 1-tensor \mathbf{X} and the 1-form \mathbf{k} :

$$\mathbf{X} = X^j v_j, \quad \mathbf{k} = k_j dx^j. \tag{6.77}$$

All tensor products will be understood as antisymmetric:

$$v_j v_k \equiv \frac{1}{2}(v_j \wedge v_k - v_k \wedge v_j), \text{ etc.} \tag{6.78}$$

This amounts to considering v_j and dx^k as anticommuting quantities. Scalar products are given by the standard contraction:

$$v_j \cdot dx^k = \delta_k^j. \tag{6.79}$$

We also revert to operator notation, dispensing with $*$ -products and writing Tr for $\rho_0 \int dx$. Finally, we simply write $\int_{(n-1)}$ for the $(n - 1)$ -dimensional s_i -integration:

$$\int_{(n-1)} \equiv \frac{1}{(2i)^{n-1}} \int_0^1 ds_1 \cdots \int_0^1 ds_{n-1} \delta \left(1 - \sum_{i=1}^{n-1} s_i \right). \tag{6.80}$$

Overall, the current in (6.76) is written as the rank- $(2n - 2)$ antisymmetric tensor \mathbf{J} ,

$$\mathbf{J} = \text{Tr} \int_{(n-1)} e^{is_1 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \cdots e^{is_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X}, \tag{6.81}$$

and its conservation is expressed by the contraction $\mathbf{k} \cdot \mathbf{J} = 0$. The contraction of \mathbf{k} acts on each \mathbf{X} in a graded fashion. Using cyclicity of trace and invariance under relabelling the s_i , this becomes

$$\mathbf{k} \cdot \mathbf{J} = (n - 1) \text{Tr} \int_{(n-1)} e^{is_1 \mathbf{k} \cdot \mathbf{X}} [\mathbf{k} \cdot \mathbf{X}, \mathbf{X}] e^{is_2 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \cdots e^{is_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X}. \tag{6.82}$$

Using the identity

$$[e^{is \mathbf{k} \cdot \mathbf{X}}, \mathbf{X}] = \int_0^s ds_1 e^{is_1 \mathbf{k} \cdot \mathbf{X}} [i \mathbf{k} \cdot \mathbf{X}, \mathbf{X}] e^{i(s-s_1) \mathbf{k} \cdot \mathbf{X}} \tag{6.83}$$

we can absorb the s_1 -integration in (6.82) and bring it to the form

$$\mathbf{k} \cdot \mathbf{J} = -\frac{1}{2} \text{Tr} \int_{(n-2)} [e^{is_2 \mathbf{k} \cdot \mathbf{X}}, \mathbf{X}] \mathbf{X} \mathbf{X} \cdots e^{is_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X}. \tag{6.84}$$

Finally, using once more the cyclicity of trace, we see that the above contraction vanishes. This proves that the tensor \mathbf{J} is conserved and, as a consequence, its dual ρ_{jk} satisfies the Bianchi identity. As in the two-dimensional case, we put

$$F_{jk}(\mathbf{k}) = \rho_{jk}(\mathbf{k}) - \omega_{jk} \delta(\mathbf{k}), \tag{6.85}$$

and recover the commuting Abelian field strength, which can, in turn, be expressed in terms of a (commutative) Abelian potential A_j .

In the above manipulations we freely used cyclicity of trace. In general this is dangerous, since the commuted operators may not be trace class. Assuming, however, that \mathbf{X} becomes asymptotically \mathbf{x} for large distances, the presence of the exponentials in the integrand ensures that this operation is permissible.

As mentioned previously, the fully symmetric ordering is not the only one that leads to an admissible ρ_{jk} . As an example, in the lowest dimensional non-trivial case $d = 4$ we can alter the ordering by splitting the commutator as

$$J^{jk} = \frac{1}{2i} \text{Tr} e^{i \mathbf{k} \cdot \mathbf{X}} [X^j, X^k] \rightarrow J_f^{jk} = -i \text{Tr} \int_0^1 ds f(s) e^{is \mathbf{k} \cdot \mathbf{X}} X^j e^{i(1-s) \mathbf{k} \cdot \mathbf{X}} X^k. \tag{6.86}$$

If $f(s) = -f(1-s)$ the above will be antisymmetric in (j, k) and conserved, as can explicitly be verified. Further, if $f(s)$ satisfies

$$\int_0^1 ds(2s - 1)f(s) = 1, \tag{6.87}$$

then (6.86) will also have the correct commutative limit. We obtain an infinity of solutions depending on a function of one variable $f(s)$. This arbitrariness reflects the fact that the Seiberg–Witten equations are not integrable and therefore the solution for $\theta = 0$ depends on the path in the θ -space taken for integrating the equations. For $d = 4$ the parameter space is a plane and the path from a given θ to $\theta = 0$ on the plane can be parametrized by a function of a single variable, just like J_f^{jk} . The various solutions are related through field redefinitions.

6.3.3. *Seiberg–Witten map in higher odd dimensions.* The situation in odd dimensions differs in that we need to specify separately the components of the conserved current in the commutative and noncommutative directions. For $d = 2n + 1$ the current is of rank- $(2n - 1)$ and it can be constructed by a procedure analogous to the even-dimensional case: we start from the expression for the total particle current j^μ (1.16) and (1.17) and introduce $2n - 2$ commutators, one less than the number which would fully saturate it to (c, \mathbf{v}) . The temporal components $J^{0j_1 \dots j_{2n-2}}$ can be expressed as a rank- $(2n - 2)$ antisymmetric spatial tensor \mathbf{J}^0 , while the spatial components $J^{j_0 j_1 \dots j_{2n-2}}$ can be expressed as a rank- $(2n - 1)$ antisymmetric tensor \mathbf{J} . Their fully ordered expressions are

$$\mathbf{J}^0 = \frac{1}{n - 1} \text{Tr} \int_{(n-1)} e^{is_1 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \dots e^{is_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X}, \tag{6.88}$$

$$\mathbf{J} = \text{Tr} \int_{(n)} e^{is_0 \mathbf{k} \cdot \mathbf{X}} D\mathbf{X} e^{is_1 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \dots e^{is_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X}. \tag{6.89}$$

The above expressions can be unified by introducing a temporal component for the field X^μ , namely $X^0 \equiv t$ (which is obviously commutative), and extending the 1-tensor \mathbf{X} also to include $X^0 v_0$. Further, we can Fourier transform in time and define $\mathbf{k} = k_\mu dx^\mu$ to include also the frequency k_0 . Then the corresponding (spacetime) $(2n - 1)$ -tensor \mathbf{J} acquires the form

$$\mathbf{J} = \int dt \text{Tr} \int_{(n)} e^{is_1 \mathbf{k} \cdot \mathbf{X}} D\mathbf{X} e^{is_2 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \dots e^{is_n \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X}. \tag{6.90}$$

X^0 is absent in $\mathbf{X} \mathbf{X}$ and, since $D X^0 = 1$, only $s_0 + s_1$ appears in the temporal component of \mathbf{J} ; integrating over s_1 reproduces the factor $1/(n - 1)$ appearing in (6.88).

The above current is obviously gauge invariant. We shall prove that it is also conserved, that is, it satisfies $\mathbf{k} \cdot \mathbf{J} = 0$. The contraction is

$$\begin{aligned} \mathbf{k} \cdot \mathbf{J} = \int dt \text{Tr} \int_{(n)} & \left\{ e^{is_1 \mathbf{k} \cdot \mathbf{X}} \mathbf{k} \cdot D\mathbf{X} e^{is_2 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \dots e^{is_n \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \right. \\ & \left. - \sum_{m=2}^n e^{is_1 \mathbf{k} \cdot \mathbf{X}} D\mathbf{X} \mathbf{X} \mathbf{X} \dots e^{is_m \mathbf{k} \cdot \mathbf{X}} [\mathbf{k} \cdot \mathbf{X}, \mathbf{X}] e^{is_{m+1} \mathbf{k} \cdot \mathbf{X}} \dots \mathbf{X} \mathbf{X} \right\} \end{aligned} \tag{6.91}$$

(with $s_{n+1} = 0$). By formula (6.83) and a similar one for the covariant time derivative, the above can be rewritten as

$$\begin{aligned} \mathbf{k} \cdot \mathbf{J} = \int dt \text{Tr} \int_{(n-1)} & \left\{ D e^{is_1 \mathbf{k} \cdot \mathbf{X}} e^{is_2 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \dots e^{is_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \right. \\ & \left. - \sum_{m=2}^{n-1} e^{is_1 \mathbf{k} \cdot \mathbf{X}} D\mathbf{X} \mathbf{X} \mathbf{X} \dots [e^{is_m \mathbf{k} \cdot \mathbf{X}}, \mathbf{X}] \mathbf{X} \mathbf{X} \dots \mathbf{X} \mathbf{X} \right\}. \end{aligned} \tag{6.92}$$

Due to the cyclicity of trace, the sum above telescopes and only the first term of the $m = 2$ commutator and the second term of the $m = n - 1$ commutator survive. Altogether we obtain

$$\begin{aligned}
 \mathbf{k} \cdot \mathbf{J} &= \int dt \operatorname{Tr} \int_{(n-1)} (D e^{i s_1 \mathbf{k} \cdot \mathbf{X}} + D \mathbf{X} \mathbf{X} + \mathbf{X} D \mathbf{X}) e^{i s_2 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \dots e^{i s_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \\
 &= \int dt \operatorname{Tr} \int_{(n-1)} D(e^{i s_1 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X}) \dots e^{i s_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \\
 &= \int dt \operatorname{Tr} \int_{(n-1)} \frac{1}{n-1} D(e^{i s_1 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \dots e^{i s_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X}) \\
 &= \int dt \frac{d}{dt} \operatorname{Tr} \int_{(n-1)} \frac{1}{n-1} e^{i s_1 \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \dots e^{i s_{n-1} \mathbf{k} \cdot \mathbf{X}} \mathbf{X} \mathbf{X} \\
 &= 0,
 \end{aligned} \tag{6.93}$$

which proves the conservation of \mathbf{J} . Its dual $\rho_{\mu\nu}$ satisfies the $(2n + 1)$ -dimensional Bianchi identity and can be used to define the commutative Abelian field strength:

$$F_{ij}(\mathbf{k}) = \rho_{ij}(\mathbf{k}) - \omega_{ij} \delta(\mathbf{k}) \tag{6.94}$$

$$F_{0i}(\mathbf{k}) = \rho_{0i}(\mathbf{k}). \tag{6.95}$$

In the above we gave separate derivations of the Seiberg–Witten map for even and odd dimensions. The two can be unified by demonstrating that each case can be obtained as a dimensional reduction of the other in one more dimension. This is treated in next section.

6.3.4. Dimensional reduction. It is quite straightforward to see that the even-dimensional Seiberg–Witten map is obtained from the $d = 2n + 1$ map by dimensional reduction. We assume a time-independent configuration in which X^j ($j = 1, \dots, 2n$) do not depend on t and A_0 vanishes. In this case $D\mathbf{X}$ vanishes and so does \mathbf{J} in (6.89); only the component \mathbf{J}^0 in (6.88) survives, reproducing the $2n$ -dimensional solution.

The reduction from a fully noncommutative $d = 2n + 2$ case to the $d = 2n + 1$ case is only slightly subtler. For concreteness, we shall take $t \equiv x^0$ to be canonically conjugate to the last dimension, call it $z \equiv x^{2n+1}$, which will be reduced; that is,

$$[t, z] = i\theta_0 \ (\theta_0 = \theta^{0,2n+1}), \quad [t, x^i] = [z, x^i] = 0 \quad (i = 1, \dots, 2n). \tag{6.96}$$

This can always be achieved with an orthogonal rotation of the x^μ . The reduced configuration consists of taking all fluid coordinates other than X^{2n+1} to be independent of x^{2n+1} and, further, the gauge potential corresponding to $z = x^{2n+1}$ to vanish. Specifically,

$$X^i = X^i(\mathbf{x}, t), \tag{6.97}$$

$$X^0 = t, \tag{6.98}$$

$$X^{2n+1} = z + \theta_0 A_0(\mathbf{x}, t). \tag{6.99}$$

With this choice the corresponding field strengths become

$$[X^i, X^j] = i\theta^{ij} + i\theta^{ik}\theta^{j\ell} \hat{F}_{k\ell} \tag{6.100}$$

$$[X^i, X^0] = 0, \tag{6.101}$$

$$[X^i, X^{2n+1}] = i\theta_0(D_0 X^i - i[X^i, A_0]) = i\theta^{ij} \theta^{2n+1,0} \hat{F}_{j0}, \tag{6.102}$$

with $\hat{F}_{\mu\nu}$ ($\mu, \nu = 0, \dots, 2n$) the field strength of a noncommutative $d = 2n + 1$ theory.

The corresponding $d = 2n + 2$ Seiberg–Witten map reduces to the $d = 2n + 1$ map. Indeed, the current \mathbf{J} in (6.81), now, is a rank- $2n$ antisymmetric tensor. When all its indices are spatial ($1, \dots, 2n$) it becomes a fully saturated topological invariant, that is, a constant; this reproduces a constant $\rho_{0,2n+1}$. When one of its indices is 0 and the rest are spatial it vanishes, leading to $\rho_{i,2n+1} = 0$. When one of its indices is $2n + 1$ and the rest are spatial it reproduces expression (6.89). Finally, when two of its indices are 0, $2n + 1$ and the rest are spatial it reproduces (6.88), recovering the full commuting $(2n + 1)$ -dimensional Abelian field strength.

We stress that the above reductions are not the most general ones. Indeed, mere invariance of the fluid configuration with respect to translations in the extra dimension does not require the vanishing of the gauge field in the corresponding direction. This means that we could choose $X^0 = t + H(\mathbf{x}, t)$ (instead of $X^0 = t$) in both $d = 2n + 1$ and $d = 2n + 2$. The corresponding reduced theory contains an extra Higgs scalar in the adjoint representation of the (noncommutative) $U(1)$ gauge group. Our Seiberg–Witten map in this situation reproduces, with no extra effort, the spacetime derivatives of a corresponding commuting ‘Higgs’ scalar.

The above complete reduction scheme ($2n + 2 \rightarrow 2n + 1 \rightarrow 2n \rightarrow \dots$) is reminiscent of the topological descent equations relevant to gauge anomalies. It is possible to consider the fluid analogues of noncommutative topological actions and the mapping of topologically non-trivial configurations [52], but we shall not consider these issues here.

A final comment. We demonstrated that gauge theory on noncommutative spaces has an effective description as a classical fluid. Other nonclassical situations may also be describable in a fluid dynamical language, thus revealing their dynamics as pertaining to a special kind of fluid. In particular, quantum mechanical many-body states could be effectively described in this fashion. This will be explored in the final section.

7. Miscellaneous topics

7.1. Quantized fluid mechanics

Eulerian fluid mechanics, even though it can be formulated as an independent dynamical system in its own right, can also be understood as a good description for the physics of a distribution of particles at large length scales. At these scales, large compared to the typical particle separations, a continuum approximation can be made and the underlying particulate nature is seldom apparent or needed. If this fluid mechanics is quantized, the result is a quantum field theory where the hydrodynamical variables of density and velocity become quantum operators. For such an analysis, it is mathematically irrelevant whether the fluid theory emerged as an approximation to the particle theory.

For quantization, the Poisson brackets (1.59), (1.62) and (1.63) are replaced by i/\hbar times a quantum commutator of the ρ and \mathbf{v} operators. There is no ordering ambiguity in the quantal version of (1.63) because ρ commutes with ω_{ij} , according to (1.62) and (1.64). Ordering has to be prescribed when the Hamiltonian (1.56) is promoted to an Hermitian operator. The kinetic term may be taken as $\frac{1}{2}v^i \rho v^i$ or as $\frac{1}{4}(\rho \mathbf{v}^2 + \mathbf{v}^2 \rho)$; the two coincide because the $[\rho, \mathbf{v}]$ -commutator is a c -number. The Heisenberg equations of motion then imply that the current contributing to the continuity equation (1.19) is the ordered Hermitian quantity

$$\mathbf{j} = \frac{1}{2}(\rho \mathbf{v} + \mathbf{v} \rho), \quad (7.1)$$

which also satisfies the quantum commutator analogues of (1.60) and (1.61). In the Euler force equation (1.20) the term non-linear in \mathbf{v} emerges upon commutation with the quantum Hamiltonian as $\frac{1}{2}(v^k \partial_k v^i + \partial_k v^i v^k)$.

It is noteworthy that the nature of the particles which constitute the fluid, in particular their statistics, does not seem to be important. For this reason, attempts to relate superfluidity to quantized hydrodynamics have been criticized [53]. However, one can see that this is clearly not the whole story by the following argument.

Consider the classical action for a (non-linear) Schrödinger field on a d -dimensional space \mathbf{R}^d ; $d = 2, 3$ are the cases of particular interest to us:

$$I = \int dt dr \left[i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2}\nabla\psi^*\cdot\nabla\psi - \mathcal{V}(\psi^*\psi) \right]. \quad (7.2)$$

With the Madelung ansatz (G.4) ($m = 1$), the action (7.2) is brought to a form appropriate for an irrotational fluid,

$$I = \int dt dr \left[\theta\dot{\rho} - \frac{1}{2}\rho(\nabla\theta)^2 - V(\rho) \right], \quad (7.3)$$

where the classical potential \mathcal{V} acquires an addition:

$$V(\rho) = \mathcal{V}(\rho) + \frac{\hbar^2}{2}(\nabla\sqrt{\rho})^2. \quad (7.4)$$

We see that the Schrödinger field theory (7.2) and fluid mechanics are equivalent with a reinterpretation of certain quantities, the particular nature of the fluid (such as its equation of state) will be characterized by the function $V(\rho)$. The Schrödinger theory (7.2) can be quantized as a fermion field or a boson field. Therefore it must be possible to quantize the fluid theory (which is after all the same thing, being a particular parametrization or choice of coordinates on the classical phase space of the theory) as either a fermion theory or a boson theory. Since the symplectic structure, given by the $\theta\dot{\rho}$ -term, is what is relevant to this consideration, the specific nature of $V(\rho)$ should not matter; it should be possible to quantize any fluid, irrespective of its dynamics, with either statistics.

This observation is certainly not new. The fluid theory is described in terms of the density ρ and the current $\mathbf{j} = \rho\nabla\theta$. The algebra of these observables, the current algebra, is given by (1.59)–(1.61). This algebra is the same whether ψ , ψ^* obey bosonic or fermionic commutation rules. Various observables of interest, such as the Hamiltonian, momentum density, etc, can be constructed in terms of ρ , \mathbf{j} . Therefore, one can take the algebra of ρ , \mathbf{j} as the starting point and construct the quantum theory in terms of a unitary irreducible representation of this algebra of operators. It is known that this algebra has many inequivalent realizations, allowing for the freedom of choosing different statistics, bosons and fermions corresponding to different representations [54]. The fluid theory can indeed carry information about the statistics of the particles which compose it.

More generally, even for a 1-component fluid, there are additional variables needed. Considering the case of three spatial dimensions as an example, the free action is

$$I_0 = - \int dt dr \left[\rho(\dot{\theta} + \alpha\dot{\beta}) + \frac{1}{2}\rho(\nabla\theta + \alpha\nabla\beta)^2 \right]. \quad (7.5)$$

The fluid velocity is given in the Clebsch form $\mathbf{v} = \nabla\theta + \alpha\nabla\beta$. The Clebsch parametrization of the velocities can be expressed as (see section 7.54, below)

$$\mathbf{v} = i\text{Tr}(\sigma_3 g^{-1}\nabla g), \quad (7.6)$$

where the group element g is an element of $SU(2)$:

$$g = \exp\left(\frac{\sigma^3}{2i}\beta\right)\exp\left(\frac{\sigma^2}{2i}\gamma\right)\exp\left(\frac{\sigma^3}{2i}\theta\right). \quad (7.7)$$

Using (7.7) with (7.6) gives the Clebsch parametrization for \mathbf{v} with $\alpha = \cos \gamma$. Even though we have the additional variables α, β , the current algebra is the same as before. Evidently, the same current algebra is realized by fluids that move with or without vortices.

Generally speaking, in two dimensions, the existence of inequivalent representations is related to the non-trivial connectivity of the space of fields. If the phase space is simply connected, it cannot support double-valued (or many-valued) wavefunctions and one does not have the possibility of different statistics. The topology of the phase space is therefore relevant to the question we are considering and one can try to characterize the inequivalent representations for the fluid current algebra along these lines. To see how this arises, it is instructive to consider the configuration space for identical particles. For N particles in two dimensions, a point in the configuration space is given by $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$, where $\mathbf{x}_i \in \mathbf{R}^2$. The identity of particles tells us that we must make the identification

$$(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, \mathbf{x}_N) \sim (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_j, \dots, \mathbf{x}_i, \dots, \mathbf{x}_N). \quad (7.8)$$

Further, we must impose the condition that the locations of the particles do not coincide, i.e., $\mathbf{x}_i \neq \mathbf{x}_j$ if $i \neq j$. The resulting configuration space \mathcal{C}_N has non-trivial connectivity; in fact, the first homotopy group is given by $\Pi_1(\mathcal{C}_N) = B_N$, where B_N is the braid group. The fact that $\Pi_1(\mathcal{C}_N) \neq 0$ shows that the configuration space can support many-valued wavefunctions. Under exchange of particles, the wavefunctions get a phase; the various phase factors for different exchanges form a unitary irreducible representation of $\Pi_1(\mathcal{C}_N)$. The phase for a single exchange can be arbitrary and so we get the possibility of arbitrary statistics. (An analogous construction can be done in three dimensions; the first homotopy group is then the permutation group S_N and we get the possibility of fermions or bosons.)

Returning to the case of two dimensions, $\mathbf{X} = (x, y)$, note that the existence of different representations is also related to the existence of a flat potential, which distinguishes between the representations. For the N -particle Heisenberg algebra, we can write a representation in terms of complex variables $w = x - iy$, $\bar{w} = x + iy$:

$$\begin{aligned} \hat{\mathbf{X}}_i &= (w_i, \bar{w}_i) \\ \hat{P}_{w_i} &= \hat{p}_{w_i} + \frac{i}{2k} \sum_{j \neq i} \frac{1}{w_i - w_j} \\ \hat{P}_{\bar{w}_i} &= \hat{p}_{\bar{w}_i} - \frac{i}{2k} \sum_{j \neq i} \frac{1}{\bar{w}_i - \bar{w}_j} \\ \hat{p}_{w_i} &= -i \frac{\partial}{\partial w_i}, \quad \hat{p}_{\bar{w}_i} = -i \frac{\partial}{\partial \bar{w}_i}. \end{aligned} \quad (7.9)$$

Here k is a parameter specifying the representation, \hat{p} gives the standard Schrödinger representation, and the summed expressions comprise the potentials, which make the representation (7.9) different from the standard one. It is easily verified that the quantities in (7.9) obey the Heisenberg algebra. (The commutator of the momenta leads to $\delta(\mathbf{x}_i - \mathbf{x}_j)$, but since we do not allow coincidence of particle locations (or the wavefunctions vanish at coincidences), this is zero.) We can write the representation (7.9) as

$$\hat{\mathbf{X}} = U^{-1} \mathbf{x} U, \quad \hat{P} = U^{-1} \hat{p} U \quad (7.10)$$

where U is given by

$$U = \exp \left(-\frac{i}{2k} \sum [\ln(w_i - w_j) - \ln(\bar{w}_i - \bar{w}_j)] \right). \quad (7.11)$$

The representation (7.9) is however not equivalent to the Schrödinger representation because U is not single valued and so does not give a unitary transformation. Note that the addition to the

expression for the momentum operators is a ‘flat potential’, in the sense that its field strength, which is given by the commutators $[\hat{P}_{w_i}, \hat{P}_{w_j}]$, is zero (with the condition of removing coincidences of particle locations). This flat potential can be written as $U^{-1}[\hat{p}, U]$, but it should be kept in mind that U is not a genuine unitary transformation.

Similar results hold in general. Let $\{\phi_a\}$ be a set of observables, which may include the identity, obeying a commutation algebra of the form

$$[\phi_a, \phi_b] = C_{ab}^c \phi_c. \tag{7.12}$$

Let $\phi_a^{(1)}$ and $\phi_b^{(2)}$ be two representations. We write $\phi_a^{(2)} = \phi_a^{(1)} + \mathcal{A}_a$. The fact that $\phi_a^{(1)}$ and $\phi_a^{(2)}$ obey the same algebra shows that \mathcal{A}_a is a flat potential, i.e.,

$$[\phi_a^{(1)}, \mathcal{A}_b] - [\phi_b^{(1)}, \mathcal{A}_a] + [\mathcal{A}_a, \mathcal{A}_b] - C_{ab}^c \mathcal{A}_c = 0. \tag{7.13}$$

If $\mathcal{A}_a = U^\dagger[\phi_a^{(1)}, U]$ for some unitary transformation U , the two representations are unitarily equivalent. However, if \mathcal{A}_a obeys the zero-curvature condition (7.13) but cannot be written as $\mathcal{A}_a = U^\dagger[\phi_a^{(1)}, U]$ for some unitary U , the two representations are inequivalent.

Clearly, the existence of inequivalent representations and the existence of a flat potential which cannot be written as $\mathcal{A}_a = U^\dagger[\phi_a^{(1)}, U]$ are related to the topology of the space of fields, or the phase space, if we are thinking in terms of canonical quantization. The U which we use form a representation of some non-trivial $\Pi_1(\mathcal{C})$; flat potentials require in general that the first cohomology group of the phase space be non-trivial. We have seen this explicitly at the level of particles. When we generalize to a field theory one can ask whether statistics can be coded in such terms. In fact, a description of the boson and fermion representations and an expression for the corresponding flat potential have been obtained [54]. It is thus possible to identify an operator, namely the flat potential \mathcal{A}_a , which can distinguish the statistics of the underlying particles. Nevertheless, the situation is not entirely satisfactory. The flat potential has been obtained only for a subspace with fixed value of the particle number N and it is in terms of the phase of the fermionic ground state wavefunction for N particles. In our opinion, this answer partially begs the question. Also, the connection to the topology of the phase space is unclear. A more direct method, in terms of the fields, would certainly be better.

Turning to the field variables ρ, \mathbf{j} , we first note that, already, the use of a group element to parametrize the velocities as in (7.7) entails making certain additional assumptions about the topology of the phase space. One has to determine on a physical grounds whether such a parametrization is justified and if so, which group should be used. For the parametrization (7.7), the phase space is given by

$$\mathcal{P} = \{\text{set of all maps } \rho : \mathbf{R}^3 \rightarrow \mathbf{R}_+, g : \mathbf{R}^3 \rightarrow G = SU(2)\}. \tag{7.14}$$

The topology of the phase space is specific to $SU(2)$: there is a compact $U(1)$ direction corresponding to θ . Since ρ and θ are canonically conjugate, the operator $U = \exp(i2\pi \int dr \rho)$ shifts θ by 2π . The compactness of θ requires that all observables be invariant under the action of U ; equivalently, the spectrum of $\int dr \rho$ must be an integer space. Thus the compactness of $U(1)$ leads to an underlying particle description, with the integer portion of $\int dr \rho$ being the number of particles N . Therefore, for hydrodynamics with an underlying particle structure, the use of a compact θ -variable is appropriate. Note also that $\int dr \rho$ is a Casimir operator for the algebra so that one can also restrict to representations with a fixed value of N . If we have only (ρ, θ) , this argument just recaptures the Schrödinger field description.

One can now try to see how the flat potentials mentioned above can arise in the field description. So far, there is no direct construction of the potential \mathcal{A}_a in terms of ρ and \mathbf{j} . In our formulation of the Clebsch parametrization, the phase space given by (7.8) does not have non-trivial first homotopy group. One might argue that this is because we have to impose

some further condition, analogous to the condition of excluding coincidence of locations at the particle level. It may be that a similar condition must be imposed, expressed in fluid variable terms, on the space of (7.14) to obtain the necessary structure for a quantum hydrodynamics, which takes account of particle statistics. It is not yet clear what such a condition would be. Also, representations of the algebra of observables are characterized, in purely algebraic terms, by the values of the Casimir operators. Thus, alternatively, one can ask the question: What are the Casimirs in terms of the fluid observables which discriminate between the different statistics? How are these Casimirs related to the topology of the phase space? As yet there is no satisfactory answer to these questions.

7.2. Fluids in intense magnetic fields

Another interesting dynamical system is a charged fluid in an intense magnetic field. This exhibits a noncommutativity, which is the fluid mechanical version of noncommuting coordinates for a point particle in a magnetic field, so intense that it effects reduction to the lowest Landau level.

7.2.1. Particle noncommutativity in the lowest Landau level. Before describing the motion of a charged fluid in an intense magnetic field, we review the story for point particles on a plane, with an external and constant magnetic field \mathbf{B} perpendicular to the plane [55]. The equation of motion for the 2-vector $\mathbf{r} = (x, y)$ is

$$m\dot{v}^i = \frac{e}{c}\epsilon^{ij}v^j B + f^i(\mathbf{r}), \quad (7.15)$$

where \mathbf{v} is the velocity $\dot{\mathbf{r}}$, and \mathbf{f} represents other forces, which we take to be derived from a potential V : $\mathbf{f} = -\nabla V$. The limit of large B is equivalent to small m . Setting the mass to zero in (7.15) leaves a first-order equation:

$$\dot{r}^i = \frac{c}{eB}\epsilon^{ij}f^j(\mathbf{r}). \quad (7.16)$$

This may be obtained by taking Poisson brackets of \mathbf{r} with the Hamiltonian

$$H_0 = V, \quad (7.17)$$

provided the fundamental brackets describe noncommuting coordinates,

$$\{r^i, r^j\} = \frac{c}{eB}\epsilon^{ij}, \quad (7.18)$$

so that

$$\dot{r}^i = \{H_0, r^i\} = \{r^j, r^i\}\partial_j V = \frac{c}{eB}\epsilon^{ij}f^j(\mathbf{r}). \quad (7.19)$$

The noncommutative algebra (7.18) and the associated dynamics can be derived in the following manner. The Lagrangian for the equation of motion (7.15) is

$$L = \frac{1}{2}m\mathbf{v}^2 + \frac{e}{c}\mathbf{v} \cdot \mathbf{A} - V. \quad (7.20)$$

When we choose the gauge $\mathbf{A} = (0, Bx)$ and set m to zero, (7.20) leaves

$$L_0 = \frac{eB}{c}x\dot{y} - V(x, y), \quad (7.21)$$

which is of the form $p\dot{q} - h(p, q)$, and one sees that $(\frac{eB}{c}x, y)$ form a canonical pair. This implies (7.18), and identifies V as the Hamiltonian.

Additionally, we give a canonical derivation of noncommutativity in the $m \rightarrow 0$ limit, starting with the Hamiltonian

$$H = \frac{\pi^2}{2m} + V. \quad (7.22)$$

H gives (7.15) upon bracketing with \mathbf{r} and \mathbf{v} , provided the following brackets hold:

$$\{r^i, r^j\} = 0 \quad (7.23)$$

$$\{\pi^i, r^i\} = \delta^{ij} \quad (7.24)$$

$$\{\pi^i, \pi^j\} = -\frac{eB}{c}\epsilon^{ij}. \quad (7.25)$$

Here π is the kinematical (non-canonical) momentum, $m\dot{\mathbf{r}}$, related to the canonical momentum \mathbf{p} by $\pi = \mathbf{p} - \frac{e}{c}\mathbf{A}$.

We wish to set m to zero in (7.22). This can only be done provided π vanishes, and we impose $\pi = 0$ as a constraint. But according to (7.25), the bracket of the constraints $\{\pi^i, \pi^j\} \equiv C^{ij} = -\frac{eB}{c}\epsilon^{ij}$ is nonzero. Hence we must introduce Dirac brackets:

$$\{O_1, O_2\}_D = \{O_1, O_2\} - \{O_1, \pi^k\}(C^{-1})^{kl}\{\pi^l, O_2\}. \quad (7.26)$$

With (7.26), any Dirac bracket involving π vanishes, so π may indeed be set to zero. But the Dirac bracket of two coordinates is now non-vanishing:

$$\{r^i, r^j\}_D = -\{r^i, \pi^k\}\frac{c}{eB}\epsilon^{kl}\{\pi^l, r^j\} = \frac{c}{eB}\epsilon^{ij}. \quad (7.27)$$

In this approach, noncommuting coordinates arise as Dirac brackets in a system constrained to lie in the lowest Landau level.

A quantum mechanical perspective on this result is as follows [56]. Let us label the degenerate quantum Landau states by (N, n) , where N labels the Landau level and n the degeneracy of that level. Consider the computation of the matrix element in the lowest ($N = 0$) Landau level of the commutator of x and y :

$$\begin{aligned} \langle 0, n|[x, y]|0, \tilde{n}\rangle &= \langle 0, n|xy|0, \tilde{n}\rangle - \langle 0, n|yx|0, \tilde{n}\rangle \\ &= \sum_{Nn'} (\langle 0, n|x|Nn'\rangle\langle Nn'|y|0, \tilde{n}\rangle - \langle 0, n|y|Nn'\rangle\langle Nn'|x|0, \tilde{n}\rangle). \end{aligned} \quad (7.28)$$

When the intermediate state sum is carried over all Landau levels, the two terms in the sum cancel each other, and we find a vanishing result; coordinates commute. Suppose however, we truncate the sum at the lowest level. A simple calculation gives a non-vanishing result, consistent with (7.18). This shows that noncommuting coordinates arise from a truncation of the Hilbert space.

(An amusing generalization determines the coordinate commutator when the first N Landau levels are retained, both in the external states and the intermediate state sum. The result is that the only non-vanishing matrix element of the $[x, y]$ -commutator is in the highest Landau level:

$$\langle N, n|[x, y]|N, n'\rangle = -\frac{i\hbar c}{eB}(N + 1)\delta(n, n'). \quad (7.29)$$

Here $\delta(n, n')$ is either the discrete or continuous δ -function, depending whether the Landau degeneracy is exhibited in a discrete or continuous manner. The result reduces to the previous, when only the lowest Landau level is kept, and also shows that as more and more states are included, the noncommutativity is pushed into ever higher states.)

7.2.2. *Field noncommutativity in the lowest Landau level.* We now turn to the equations of a charged fluid with density ρ and mass parameter m (introduced for dimensional reasons) moving on a plane with velocity \mathbf{v} in an external magnetic field perpendicular to the plane. ρ and \mathbf{v} are functions of t and \mathbf{r} and give an Eulerian description of the fluid. The equations that are satisfied are the continuity equation (1.65), and the Euler equation (1.66), which acquires an additional term, beyond the force term, due to the magnetic interaction (compare (5.46) reduced to the Abelian case):

$$m\dot{v}^i + m\mathbf{v} \cdot \nabla v^i = \frac{e}{c}\epsilon^{ij}v^j B + f^i. \quad (7.30)$$

Here \mathbf{f} describes additional forces, e.g. $-\frac{1}{\rho}\nabla P$ where P is pressure. We shall take the force to be derived from a potential in the form

$$\mathbf{f}(\mathbf{r}) = -\nabla \frac{\delta}{\delta\rho(\mathbf{r})} \int d^2r V. \quad (7.31)$$

(For isentropic systems, the pressure is only a function of ρ . Here we allow more general dependence of V on ρ (e.g. non-locality or dependence on derivatives of ρ .)

The relevant equations follow by bracketing ρ and \mathbf{v} with the Hamiltonian

$$H = \int d^2r \left(\rho \frac{\pi^2}{2m} + V \right), \quad (7.32)$$

provided that fundamental brackets are taken as

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = 0 \quad (7.33)$$

$$\{\pi(\mathbf{r}), \rho(\mathbf{r}')\} = \nabla\delta(\mathbf{r} - \mathbf{r}') \quad (7.34)$$

$$\{\pi^i(\mathbf{r}), \pi^j(\mathbf{r}')\} = -\epsilon^{ij} \frac{1}{\rho} \left(m\omega(\mathbf{r}) + \frac{eB}{c} \right) \delta(\mathbf{r} - \mathbf{r}') \quad (7.35)$$

where $\epsilon^{ij}\omega(\mathbf{r})$ is the vorticity $\partial_i v^j - \partial_j v^i$, and $\pi = m\mathbf{v}$. These are just the rescaled (by m) versions of (1.59), (1.62) and (1.63), except that the last is modified to include the constant magnetic field.

We now consider a strong magnetic field and take the limit $m \rightarrow 0$, which is equivalent to large B . Equations (7.30) and (7.31) reduce to

$$v^j = -\frac{c}{eB}\epsilon^{ij} \frac{\partial}{\partial r^j} \frac{\delta}{\delta\rho(\mathbf{r})} \int d^2r V. \quad (7.36)$$

Combining this with the continuity equation gives the equation for the density ‘in the lowest Landau level’:

$$\dot{\rho}(\mathbf{r}) = \frac{c}{eB} \frac{\partial}{\partial r^i} \rho(\mathbf{r}) \epsilon^{ij} \frac{\partial}{\partial r^j} \frac{\delta}{\delta\rho(\mathbf{r})} \int d^2r V. \quad (7.37)$$

(For the right-hand side not to vanish, V must not be solely a function of ρ .)

The equation of motion (7.37) can be obtained by bracketing with the Hamiltonian

$$H_0 = \int d^2r V, \quad (7.38)$$

provided the charge density bracket is non-vanishing, showing noncommutativity of ρ [50]:

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = -\frac{c}{eB}\epsilon^{ij} \partial_i \rho(\mathbf{r}) \partial_j \delta(\mathbf{r} - \mathbf{r}'). \quad (7.39)$$

H_0 and this bracket may be obtained from (7.32) and (7.33)–(7.35) with the same Dirac procedure presented for the particle case: we wish to set m to zero in (7.32); this is possible only if π is constrained to vanish. But the bracket of the π is non-vanishing, even at $m = 0$, because $B \neq 0$. Thus at $m = 0$ we posit the Dirac brackets

$$\{O_1(\mathbf{r}_1), O_2(\mathbf{r}_2)\}_D = \{O_1(\mathbf{r}_1), O_2(\mathbf{r}_2)\} - \int d^2\mathbf{r}'_1 d^2\mathbf{r}'_2 \{O_1(\mathbf{r}_1), \pi^i(\mathbf{r}'_1)\} (C^{-1})^{ij}(\mathbf{r}'_1, \mathbf{r}'_2) \{\pi^j(\mathbf{r}'_2), O_2(\mathbf{r}_2)\}, \quad (7.40)$$

where C^{ij} is the bracket in (7.35) (at $m = 0$) so that

$$(C^{-1})^{ij}(\mathbf{r}_1, \mathbf{r}_2) = \frac{c}{eB} \epsilon^{ij} \rho(\mathbf{r}_1) \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (7.41)$$

Hence Dirac brackets with π vanish, and the Dirac bracket of densities is non-vanishing as in (7.39):

$$\begin{aligned} \{\rho(\mathbf{r}), \rho(\mathbf{r}')\}_D &= -\frac{c}{eB} \int d^2\mathbf{r}'' \{\rho(\mathbf{r}), \pi^i(\mathbf{r}'')\} \rho(\mathbf{r}'') \epsilon^{ij} \{\pi^j(\mathbf{r}''), \rho(\mathbf{r}')\} \\ &= -\frac{c}{eB} \epsilon^{ij} \partial_i \rho(\mathbf{r}) \partial_j \delta(\mathbf{r} - \mathbf{r}'). \end{aligned} \quad (7.42)$$

The ρ -bracket enjoys a more appealing expression in momentum space. Upon defining

$$\tilde{\rho}(\mathbf{p}) = \int d^2\mathbf{r} e^{i\mathbf{p}\cdot\mathbf{r}} \rho(\mathbf{r}) \quad (7.43)$$

we find

$$\{\tilde{\rho}(\mathbf{p}), \tilde{\rho}(\mathbf{q})\}_D = -\frac{c}{eB} \epsilon^{ij} p^i q^j \tilde{\rho}(\mathbf{p} + \mathbf{q}). \quad (7.44)$$

The brackets (7.39), (7.44) give the algebra of area-preserving diffeomorphisms. Indeed (7.44) follows from the full diffeomorphism algebra (1.61) with the identification that ρ in (7.44) is related to j^i in (1.61) by $\rho = \epsilon^{ij} \partial_i j^j$.

The form of the charge density bracket (7.39), (7.42), (7.44) can be understood by reference to the particle substructure for the fluid. Take as in (1.12)

$$\rho(\mathbf{r}) = \sum_n \delta(\mathbf{r} - \mathbf{r}_n), \quad (7.45)$$

where n labels the individual particles. The coordinates of each particle satisfy the non-vanishing bracket (7.18). Then the $\{\rho(\mathbf{r}), \rho(\mathbf{r}')\}$ bracket takes the form described above.

7.2.3. Quantization. Quantization before the reduction to the lowest Landau level is straightforward. For the particle case (7.23)–(7.25) and for the fluid case (7.33)–(7.35) we replace brackets with i/\hbar times commutators. After reduction to the lowest Landau level we do the same for the particle case thereby arriving at the ‘Peierls substitution’, which states that the effect of an impurity (V in (7.20)) on the lowest Landau energy level can be evaluated to lowest order by viewing the (x, y) arguments of V as noncommuting variables.

However, for the fluid case quantization presents a choice. On the one hand, we can simply promote the brackets (7.39), (7.42), (7.44) to a commutator by multiplying by i/\hbar :

$$[\rho(\mathbf{r}), \rho(\mathbf{r}')] = i\hbar \frac{c}{eB} \epsilon^{ij} \partial_i \rho(\mathbf{r}') \partial_j \delta(\mathbf{r} - \mathbf{r}') \quad (7.46)$$

$$[\tilde{\rho}(\mathbf{p}), \tilde{\rho}(\mathbf{q})] = i\hbar \frac{c}{eB} \epsilon^{ij} p^i q^j \tilde{\rho}(\mathbf{p} + \mathbf{q}). \quad (7.47)$$

Alternatively, we can adopt expression (7.45), for the operator $\rho(\mathbf{r})$, where the \mathbf{r}_n now satisfy the noncommutative algebra,

$$[r_n^i, r_{n'}^j] = -i\hbar \frac{c}{eB} \epsilon^{ij} \delta_{nn'} \quad (7.48)$$

and calculate the ρ -commutator as a derived quantity.

However, once \mathbf{r}_n is a noncommuting operator, functions of \mathbf{r}_n , even δ -functions, have to be ordered. We choose the Weyl ordering, which is equivalent to defining the Fourier transform as

$$\tilde{\rho}(\mathbf{p}) = \sum_n e^{i\mathbf{p}\cdot\mathbf{r}_n}. \quad (7.49)$$

With the help of (7.48) and the Baker–Hausdorff lemma, we arrive at the ‘trigonometric algebra.’ [57]

$$[\tilde{\rho}(\mathbf{p}), \tilde{\rho}(\mathbf{q})] = 2i \sin\left(\frac{\hbar c}{2eb} \epsilon^{ij} p^i q^j\right) \tilde{\rho}(\mathbf{p} + \mathbf{q}). \quad (7.50)$$

This reduces to (7.47) for small \hbar .

This form for the commutator, (7.50), is connected to a Moyal $*$ -product in the following fashion. For an arbitrary c -number function $f(\mathbf{r})$ define

$$\langle f \rangle = \int d^2r \rho(\mathbf{r}) f(\mathbf{r}) = \frac{1}{(2\pi)^2} \int d^2p \tilde{\rho}(\mathbf{p}) \tilde{f}(-\mathbf{p}). \quad (7.51)$$

Multiplying (7.50) by $\tilde{f}(-\mathbf{p})\tilde{g}(-\mathbf{q})$ and integrating gives

$$[\langle f \rangle, \langle g \rangle] = \langle h \rangle, \quad (7.52)$$

with

$$h(\mathbf{r}) = (f * g)(\mathbf{r}) - (g * f)(\mathbf{r}) \quad (7.53)$$

where the $*$ -product is defined in (6.5). Note, however, that only the commutator is mapped into the $*$ -commutator. The product $\langle f \rangle \langle g \rangle$ is not equal to $\langle f * g \rangle$.

The lack of consilience between (7.47) and (7.50) is an instance of the Groenewold–VanHove theorem which establishes the impossibility of taking over into quantum mechanics all classical brackets. Equations (7.50)–(7.53) explicitly exhibit the physical occurrence of the $*$ -product for fields in a strong magnetic background.

7.3. Fluids and ferromagnets

A quantum system that has some similarity in its description to fluid mechanics is the ferromagnet [58]. The requirements that we put on the ferromagnetic theory are that it gives rise to a local spin algebra (compare (5.20))

$$[S^a(\mathbf{r}), S^b(\mathbf{r}')] = i\hbar \epsilon_{abc} S^c(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{r}') \quad (7.54)$$

and that some Hamiltonian H encode the specific nature of the ferromagnet. Thus we expect the action to be

$$I = I_0 - \int dt H. \quad (7.55)$$

Here I_0 involves the canonical 1-form that leads to (7.54), where the S^a are definite functions of the canonical variables.

7.3.1. *Spin algebra.* In our treatment of non-Abelian fluids we obtained the Poisson bracket version of (7.54), i.e. (5.20), by starting from (D.1) specialized to $SU(2)$: g , an element of $SU(2)$, $K = \sigma^3/2i$,

$$S_a T^a = g K g^{-1}. \quad (7.56)$$

However, for the ferromagnet, there is no place for a density variable ρ ; rather we should set ρ in (D.1) to a constant and posit for the canonical portion of the Lagrange density

$$\mathcal{L}_0 = -i \operatorname{Tr} \sigma^3 g^{-1} \dot{g}. \quad (7.57)$$

The canonical variables are three in number, corresponding to the three parameters specifying the $SU(2)$ group element. Consequently the 2-form involves a singular matrix (3×3 and antisymmetric) with no inverse. Nevertheless, one can overcome this obstacle and conclude that (7.54) continues to hold with S_α defined by (7.55).

This is achieved with the following steps. With the notation of appendix D, we determine that the \mathcal{L}_0 reads (compare (D.2 and (D.3))

$$\mathcal{L}_0 = \dot{\varphi} C^a{}_b S_b. \quad (7.58)$$

The canonical 1-form has components $a_a = C^a{}_b S_b$ (compare (D.10)) and the 2-form reads

$$\frac{\delta}{\delta \varphi^a} a_b - \frac{\delta}{\delta \varphi^b} a_a \equiv f_{ab}, \quad f_{ab}(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') C^c{}_a C^d{}_b S^e \varepsilon_{cde}. \quad (7.59)$$

Consider now the left translation of g : $\delta g = \epsilon^a \sigma^a g/2i$, or equivalently $\delta \varphi_a = -\epsilon^b c^b{}_a$, which leads to $\delta S^a = \varepsilon_{abc} \epsilon^b S^c$. The generator of this transformation satisfies, according to (A.2),

$$\frac{\delta G}{\delta \varphi_a} = -\epsilon^f c^f{}_b C^c{}_b C^d{}_a S^e \varepsilon_{cde} = -\epsilon^c C^d{}_a S^e \varepsilon_{cde}. \quad (7.60)$$

This equation is solved by

$$G = - \int d\mathbf{r} S^a \epsilon^a. \quad (7.61)$$

From the general theory, (A.17), we knew that Poisson bracketing with G generates the above transformation on any function of the phase space variables, in particular on S^b :

$$\varepsilon_{abc} \epsilon^b(\mathbf{r}') S^c(\mathbf{r}') = \{G, S^b(\mathbf{r}')\} = \left\{ \int d\mathbf{r} \epsilon^a(\mathbf{r}) S^a(\mathbf{r}), S^b(\mathbf{r}') \right\}. \quad (7.62)$$

Stripping away the arbitrary parameter $\epsilon^a(\mathbf{r})$ leads to a Poisson bracket, which gives (7.54) upon quantization.

It is instructive to obtain this result by applying the procedures in appendix A.2, relevant to singular 2-forms. Note that S^a satisfies

$$\frac{\delta}{\delta \varphi_b} S^a = c^a{}_m f^m{}_b. \quad (7.63)$$

Consequently, it follows from (A.10) that the admissibility criterion (A.13) is obeyed. The bracket of local spins reads

$$\{S^a(\mathbf{r}), S^b(\mathbf{r}')\} = \int d\mathbf{r}'' d\mathbf{r}''' \frac{\delta}{\delta \varphi_c(\mathbf{r}')} S^a(\mathbf{r}) f^{cd}(\mathbf{r}', \mathbf{r}'') \frac{\delta}{\delta \varphi_d(\mathbf{r}'')} S^b(\mathbf{r}'). \quad (7.64)$$

It follows from (7.58), (A.10), (A.11) and (7.61) that (7.62) reduces to the Poisson bracket version of (7.54).

7.3.2. *Momentum density algebra.* The above shows that the ferromagnet in the continuum approximation, may be considered, as far as the canonical structure is concerned, as fluid mechanics with a constant density ρ . It has been known for some time that there is difficulty in defining a momentum density for the ferromagnet [59]. The momentum density must generate the coordinate transformation $\delta \mathbf{r} = -\boldsymbol{\delta}(\mathbf{r})$, $\delta \varphi_a = \boldsymbol{\delta} \cdot \nabla \varphi_a$,

$$v^{\varphi_a} = -\boldsymbol{\delta} \cdot \nabla \varphi_a \quad (7.65)$$

and the generator G should solve (see (A.17), (D.12) and (D.13b))

$$\begin{aligned} \frac{\delta G}{\delta \varphi_a} &= \int d\mathbf{r} \boldsymbol{\delta} \cdot \nabla \varphi_b C^c{}_b C^d{}_a S^e \varepsilon_{cde} \\ &= \frac{\delta}{8\varphi_a} \int d\mathbf{r} \boldsymbol{\delta} \cdot \text{Tr} \nabla g g^{-1} S + i \int d\mathbf{r} \nabla \cdot \boldsymbol{\delta} \text{Tr} \frac{\delta g}{\delta \varphi_a} g^{-1} S. \end{aligned} \quad (7.66)$$

The second equality follows from the first with (D.2) and (7.56). The last term prevents equation (7.65) from being integrable. Therefore, a generator of local translations, i.e. a momentum density, cannot be defined, except in the case that $\nabla \cdot \boldsymbol{\delta} = 0$. Translations with transverse $\boldsymbol{\delta}$ correspond to volume-preserving transformations; and are canonically implemented with the generator

$$G(\boldsymbol{\delta}) = \int d\mathbf{r} \boldsymbol{\delta} \cdot \text{Tr} \nabla g g^{-1} S. \quad (7.67)$$

They include total momentum and orbital angular momentum.

It is instructive to see why the procedures in appendix A.2 fail to produce a proper generator, which by Noether's theorem for the Lagrange density (7.57) is

$$\mathcal{P} = a_a(\varphi) \nabla \varphi_a = 2 \text{Tr} \nabla g g^{-1} S. \quad (7.68)$$

The bracket of the quantities

$$\{\mathcal{P}^i(\mathbf{r}), \mathcal{P}^j(\tilde{\mathbf{r}})\} = \int d\mathbf{r}' d\mathbf{r}'' \left(\frac{\delta}{\delta \varphi_a(\mathbf{r}')} \mathcal{P}^i(\mathbf{r}') \right) f^{ab}(\mathbf{r}', \mathbf{r}'') \left(\frac{\delta}{\delta \varphi_b(\mathbf{r}'')} \mathcal{P}^j(\tilde{\mathbf{r}}) \right) \quad (7.69a)$$

is evaluated with the help of the projected inverse of f^{ab} , which satisfies (A.11). The result is

$$\begin{aligned} \{\mathcal{P}^i(\mathbf{r}), \mathcal{P}^j(\mathbf{r}')\} &= \mathcal{P}^i(\mathbf{r}') \partial_i \delta(\mathbf{r} - \mathbf{r}') + \mathcal{P}^j(\mathbf{r}) \partial_j \delta(\mathbf{r} - \mathbf{r}') \\ &\quad - \Delta^j(\mathbf{r}') \partial_i \delta(\mathbf{r} - \mathbf{r}') - \Delta^i(\mathbf{r}) \partial_j \delta(\mathbf{r} - \mathbf{r}'), \end{aligned} \quad (7.69b)$$

where

$$\Delta \equiv a_m P_a^m \nabla \varphi_a, \quad (7.70)$$

with P_a^m being the projector on the zero modes of f_{ab} . The usual momentum density algebra should not contain the Δ modification (7.69), (7.70) (compare (1.61)). This modification also prevents the quantities $\int d\mathbf{r} \boldsymbol{\delta}(\mathbf{r}) \mathbf{P}(\mathbf{r})$ from realizing the full diffeomorphism algebra (1.85)–(1.87), except in the restricted case of volume-preserving diffeomorphisms $\nabla \cdot \boldsymbol{\delta} = 0$. Finally, we remark that with non-vanishing Δ the acceptability condition (A.13) is not met, and the bracket (7.70) does not satisfy the Jacobi identity.

A solution to all these problems with implementing arbitrary diffeomorphisms is found through the connection to fluid mechanics, where the ferromagnetic phase space is enlarged to include ρ as in (D1) [58]. Then the 2-form is non-singular, the symplectic structure and conventional Poisson brackets exist (see (D.5)). Either equation (A.17) or Noether's theorem gives the generator of an arbitrary diffeomorphism, namely the momentum density, as

$$\mathcal{P} = 2\rho \text{Tr} \nabla g g^{-1} S, \quad (7.71)$$

which satisfies the proper algebra (1.61), or (7.69b) without the Δ spoiler. The ρ -extended phase space can be reduced, in the Dirac sense, by imposing the first class constraint $\rho = \text{constant}$. This will achieve the ferromagnet's phase space.

7.4. *Non-Abelian Clebsch parametrization (or, non-Abelian Chern–Simons term as a surface integral—a holographic presentation)*

In appendix B we describe the Clebsch parametrization (B.1) for a three-dimensional vector potential, which casts the Abelian Chern–Simons density (B.11) into a total derivative form (B.11), so that its three-dimensional volume integral obtains its value from the two-dimensional surface that bounds the integration volume. One may pose an analogous question about a non-Abelian gauge potential: how should it be parametrized so that the non-Abelian Chern–Simons term

$$\text{CS}(A) = \varepsilon^{ijk} \left(A_i^a \partial_j A_j^a + \frac{1}{3} f_{abc} A_i^a A_j^b A_k^c \right) = -\varepsilon^{ijk} \text{Tr} \left(\frac{1}{2} A_i \partial_j A_k + \frac{1}{3} A_i A_j A_k \right) \quad (7.72)$$

becomes a total derivative, and its integral becomes a surface term? (In the second equality we use a matrix representation of the Lie algebra: $\mathbf{A} = \mathbf{A}^a T^a$, $\text{Tr} T^a T^b = -\delta^{ab}/2$.) This question has been answered in different ways by physicists and by mathematicians. We shall here present both constructions, but first we explain the reason for the difference.

Consider $\text{CS}(A)$ as a 3-form on an arbitrary manifold,

$$\text{CS}(A) = A^a dA^a + \frac{1}{3} f_{abc} A^a A^b A^c \quad A_i^a \equiv A^a dx^i. \quad (7.73)$$

(In our form notation, we suppress the wedge product.) It follows that $d \text{CS}(A)$ is a 4-form:

$$d \text{CS}(A) = F^a F^a \quad F^a = \frac{1}{2} F_{ij}^a dx^i dx^j. \quad (7.74)$$

If the Chern–Simons term is a total derivative, $\text{CS}(A) = d\Theta$, the $d \text{CS}(A) = 0$. So the possibility of expressing the Chern–Simons as a total derivative requires that $F^a F^a$ vanish. The ‘physics’ approach to the problem achieves vanishing of the 4-form $d \text{CS}(A) = F^a F^a$ by working on a three-dimensional manifold, which does not support 4-forms. (The Abelian Clebsch parametrization (B.1) is given in 3-space!) The ‘mathematics’ approach remains with the four-dimensional space, viewed as a Kähler manifold, but requires that certain components of F vanish, see below.

7.4.1. *Total derivative form for the Chern–Simons term in 3-space.* In appendix B, equations (B.5)–(B.10), we gave an analytic/geometric construction of the Abelian Clebsch parametrization for a vector potential. However, there is another approach to this Abelian problem, which contains clues for the non-Abelian generalization. So we re-analyse the Abelian case.

The alternate method for constructing the Clebsch parametrization for an Abelian potential relies on projecting the potential from a non-Abelian one, specifically, one for $SU(2)$. We consider an $SU(2)$ group element g and a pure gauge $SU(2)$ gauge field, whose matrix-valued 1-form is

$$g^{-1} dg = V^a \frac{\sigma^a}{2i}, \quad (7.75)$$

where σ^a are Pauli matrices. It is known that

$$\text{Tr}(g^{-1} dg)^3 = -\frac{1}{4} \varepsilon_{abc} V^a V^b V^c = -\frac{3}{2} V^1 V^2 V^3 \quad (7.76)$$

is a total derivative; indeed its spatial integral measures the winding number of the gauge function g [61]. Since V^a is a pure gauge, we have

$$dV^a = -\frac{1}{2} \varepsilon_{abc} V^b V^c, \quad (7.77)$$

so that if we define an Abelian gauge potential A by selecting one $SU(2)$ component of (7.75) (say the third) $A = V^3$, the Abelian Chern–Simons density for A is a total derivative, as is seen from the chain of equations that relies on (7.76) and (7.77),

$$A dA = V^3 dV^3 = -V^1 V^2 V^3 = \frac{2}{3} \text{Tr}(g^{-1} dg)^3, \quad (7.78)$$

and concludes with an expression known to be a total derivative. Of course $A = V^3$ is not an Abelian pure gauge.

Note that g depends on three arbitrary functions, the three $SU(2)$ local gauge functions. Hence V^3 enjoys sufficient generality to represent the three-dimensional vector A . Moreover, since a Abelian Chern–Simons density is given by $\text{Tr}(g^{-1} dg)^3$, which is a total derivative, a Clebsch parametrization for A is easily constructed. We also observe that when the $SU(2)$ group element g has non-vanishing winding number, the resultant Abelian vector possesses a non-vanishing Chern–Simons integral, that is, nonzero magnetic helicity. Specifically, the example of the Clebsch parametrized gauge potential in (B.15), (B.18) is obtained by projecting onto the third direction of a pure gauge $SU(2)$ potential constructed from the group element $g = \exp(\sigma^a/2i)\hat{r}^a a(r) 2A = i \text{Tr} \sigma^3 g^{-1} dg$. An even more direct example is given by the $SU(2)$ group element $g = \exp(\frac{\sigma^3}{2i}\beta) \exp(\frac{\sigma^2}{2i}\gamma) \exp(\frac{\sigma^3}{2i}\theta)$. Then $A = d\theta + \cos \gamma d\beta$.

Now we turn to the non-Abelian problem, which we formulate in the following way: for a given group H , how can one construct a potential A^a such that the non-Abelian Chern–Simons integrand $\text{CS}(A)$ is a total derivative?

In the solution that we present [62], the ‘total derivative’ form for the Chern–Simons density of A^a is achieved in two steps. The parametrization, which we find, directly leads to an Abelian form of the Chern–Simons density,

$$A^a dA^a + \frac{1}{3} f_{abc} A^a A^b A^c = \gamma d\gamma, \quad (7.79)$$

for some γ . Then Darboux’s theorem (or usual fluid dynamical theory) ensures that γ can be presented in Clebsch form, so that $\gamma d\gamma$ is explicitly a total derivative.

We begin with a pure gauge $g^{-1} dg$ in some non-Abelian group G (called the Ur-group) whose Chern–Simons integral coincides with the winding number of g :

$$W(g) = \frac{1}{16\pi^2} \int d^3r \text{CS}(g^{-1} dg) = \frac{1}{24\pi^2} \int d^3r \text{Tr}(g^{-1} dg)^3. \quad (7.80)$$

We consider a normal subgroup $H \subset G$, with generators T^a , and construct a non-Abelian gauge potential for H by projection:

$$A^a \propto \text{Tr}(T^a g^{-1} dg). \quad (7.81)$$

Within H , this is not a pure gauge. We determine the group structure that ensures the Chern–Simons 3-form of A^a to be proportional to $\text{Tr}(g^{-1} dg)^3$. Consequently, the constructed non-Abelian gauge fields, belonging to the group H , carry quantized Chern–Simons number. Moreover, we describe the properties of the Ur-group G that guarantee that the projected potential A^a enjoys sufficient generality to represent an arbitrary potential in H .

Since $\text{Tr}(g^{-1} dg)^3$ is a total derivative for an arbitrary group [61] (although this fact cannot in general be expressed in finite terms) our construction ensures that the form of A^a , which is achieved through the projection (7.81), produces a ‘total derivative’ expression (in the limited sense indicated above) for its Chern–Simons density.

Conditions on the Ur-group G , which we take to be compact and semi-simple, are as follows. First of all G has to be so chosen that it has sufficient number of parameters to make $\text{Tr}(T^a g^{-1} dg)$ a generic potential for H . Since we are in three dimensions, an H -potential A_i^a has $3 \times \dim H$ independent functions; so a minimal requirement will be

$$\dim G \geq 3 \dim H. \quad (7.82)$$

Secondly, we require that the H -Chern–Simons form for A^a coincide with that of $g^{-1} dg$. As we shall show in a moment, this is achieved if G/H is a symmetric space. In this case, if we split the Lie algebra of G into the H -subalgebra spanned by T^a , $a = 1, \dots, \dim H$, and

the orthogonal complement spanned by S^A , $A = 1, \dots, (\dim G - \dim H)$, the commutation rules are of the form

$$[T^a, T^b] = f_{abc} T^c, \tag{7.83a}$$

$$[T^a, S^A] = h^{aAB} S^B, \tag{7.83b}$$

$$[S^A, S^B] = N h^{aAB} T^a. \tag{7.83c}$$

$(h^a)^{AB}$ form a (possibly reducible) representation of the H -generators T^a . The constant N depends on normalizations. More explicitly, if the structure constants for the Ur-group G are named \tilde{f}_{abc} , $\{a, b, c\} = 1, \dots, \dim G$, then the conditions (7.83a–7.83c) require that \tilde{f}_{abc} vanishes whenever an odd number of indices belongs to the orthogonal complement labelled by A, B, \dots . Moreover, f_{abc} are taken to be the conventional structure constants for H and this may render them proportional to (rather than equal to) \tilde{f}_{abc} .

We define the traces of the generators by

$$\text{Tr}(T^a T^b) = -N_1 \delta^{ab}, \quad \text{Tr}(S^A S^B) = -N_2 \delta^{AB} \quad \text{Tr}(T^a S^A) = 0. \tag{7.84}$$

We can evaluate the quantity $\text{Tr}[S^A, S^B] T^a = \text{Tr} S^A [S^B, T^a]$ using the commutation rules. This immediately gives the relation $N_1 N = N_2$.

Expanding $g^{-1} dg$ in terms of generators, we write

$$g^{-1} dg = T^a A^a + S^A \alpha^A, \tag{7.85}$$

which defines the H -potential A^a . Equivalently

$$A^a = -\frac{1}{N_1} \text{Tr}(T^a g^{-1} dg). \tag{7.86}$$

From $d(g^{-1} dg) = -g^{-1} dg g^{-1} dg$, we get the Maurer–Cartan relations

$$F^a \equiv dA^a + \frac{1}{2} f_{abc} A^b A^c = -\frac{N}{2} h^{aAB} \alpha^A \alpha^B, \quad d\alpha^A + h^{aBA} A^a \alpha^B = 0. \tag{7.87}$$

Using these results, the following chain of equations shows that the Chern–Simons 3-form for the H -gauge group is proportional to $\text{Tr}(g^{-1} dg)^3$:

$$\begin{aligned} \frac{1}{16\pi^2} \left(A^a dA^a + \frac{1}{3} f^{abc} A^a A^b A^c \right) &= \frac{1}{48\pi^2} (A^a dA^a + 2 A^a F^a) \\ &= \frac{1}{48\pi^2} (A^a dA^a - N h^{aAB} A^a \alpha^A \alpha^B) \\ &= \frac{1}{48\pi^2} (A^a dA^a + N d\alpha^A \alpha^A) \\ &= -\frac{1}{48\pi^2} \left(\frac{1}{N_1} \text{Tr} A dA + \frac{N}{N_2} \text{Tr} d\alpha \alpha \right) \\ &= -\frac{1}{48\pi^2 N_1} \text{Tr}(A dA + \alpha d\alpha) \\ &= -\frac{1}{48\pi^2 N_1} \text{Tr} g^{-1} dg d(g^{-1} dg) \\ &= \frac{1}{48\pi^2 N_1} \text{Tr}(g^{-1} dg)^3. \end{aligned} \tag{7.88}$$

In the above sequence of manipulations, we have used the Maurer–Cartan relations (7.87), which rely on the symmetric space structure of (7.83a)–(7.83c), and the trace relations (7.84), along with $N_1 N = N_2$.

We thus see that $\int \text{CS}(A)$ is indeed the winding number of the configuration $g \in G$. Since $\text{Tr}(g^{-1} dg)^3$ is a total derivative locally on G , the potential (7.86), with the symmetric

space structure of (7.83a)–(7.83c), does indeed fulfil the requirement of making $\text{CS}(A)$ a total derivative. It is therefore appropriate to call our construction (7.86) a ‘non-Abelian Clebsch parametrization’.

In explicit realizations, given a gauge group of interest H , we need to choose a group G such that the conditions (7.82), (7.83a)–(7.83c) hold. In general this is not possible. However, one can proceed recursively. Let us suppose that the desired result has been established for a group, which we call H_2 . Then we form $H \subset G$ obeying (7.83a)–(7.83c) as $H = H_1 \times H_2$, where H_1 is the gauge group of interest, satisfying $\dim G \geq 3 \dim H_1$. For this choice of H , the result (7.88) becomes

$$\text{CS}(H_1) + \text{CS}(H_2) = \frac{1}{48\pi^2 N_1} \text{Tr}(g^{-1} dg)^3. \quad (7.89)$$

But since $\text{CS}(H_2)$ is already known to be a total derivative, (7.89) shows the desired result: $\text{CS}(H_1)$ is a total derivative.

With $SU(2)$ as the Ur-group and $H = U(1)$ or $SO(2)$, we achieve Clebsch parametrization for an Abelian potential, as explained in (7.75)–(7.78) ($T^a = \frac{\sigma^a}{2i}$; $S^A = \{\frac{\sigma^2}{2i}, \frac{\sigma^3}{2i}\}$).

For an explicit working out of a non-Abelian case, we consider a potential for $SU(2) \approx O(3)$, which possesses nine independent functions. We take $G = O(5)$, $H = O(3) \times O(2)$. We consider the four-dimensional spinorial representation of $O(5)$. With the generators normalized by $\text{Tr}(T^a T^b) = -\delta^{ab}$, the Lie algebra generators of $O(5)$ are given by matrices of Pauli matrices:

$$\begin{aligned} T^a &= \frac{1}{2i} \begin{pmatrix} \sigma^a & 0 \\ 0 & \sigma^a \end{pmatrix} \\ T^0 &= \frac{1}{2i} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \\ S^A &= \frac{1}{i\sqrt{2}} \begin{pmatrix} 0 & 0 \\ \sigma^A & 0 \end{pmatrix} \quad \tilde{S}^A = \frac{1}{i\sqrt{2}} \begin{pmatrix} 0 & \sigma^A \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (7.90)$$

Here T^a generate $O(3)$, with the conventional structure constants ϵ_{abc} , and T^0 is the generator of $O(2)$. S, \tilde{S} are the coset generators.

A general group element in $O(5)$ can be written in the form $g = Mhk$ where $h \in O(3)$, $k \in O(2)$, and

$$M = \frac{1}{\sqrt{1 + \bar{\mathbf{w}} \cdot \mathbf{w} - \frac{1}{4}(\mathbf{w} \times \bar{\mathbf{w}})^2}} \begin{pmatrix} I - \frac{i}{2}(\mathbf{w} \times \bar{\mathbf{w}}) \cdot \sigma & -\mathbf{w} \cdot \sigma \\ \bar{\mathbf{w}} \cdot \sigma & I + \frac{i}{2}(\mathbf{w} \times \bar{\mathbf{w}}) \cdot \sigma \end{pmatrix}, \quad (7.91)$$

where w^a is a complex three-dimensional vector, with the bar denoting complex conjugation. $\mathbf{w} \cdot \bar{\mathbf{w}} = w^a \bar{w}^a$ and $(\mathbf{w} \times \bar{\mathbf{w}})^a = \epsilon_{abc} w^b \bar{w}^c$. The general $O(5)$ group element contains ten independent real functions. These are collected as six from M (in the three complex functions w^a), three in h , and one in k .

The $O(3)$ gauge potential given by $-\text{Tr}(I^a g^{-1} dg)$ reads

$$\begin{aligned} A^a &= R^{ab}(h)a^b + (h^{-1} dh)^a \\ a^a &= \frac{1}{1 + \mathbf{w} \cdot \bar{\mathbf{w}} - \frac{1}{4}(\mathbf{w} \times \bar{\mathbf{w}})^2} \left\{ \frac{1}{2} w^a d\bar{\mathbf{w}} \cdot (\mathbf{w} \times \bar{\mathbf{w}}) + \bar{w}^a d\mathbf{w} \cdot (\bar{\mathbf{w}} \times \mathbf{w}) \right. \\ &\quad \left. + \epsilon_{abc} (dw^b \bar{w}^c - w^b d\bar{w}^c) \right\}, \end{aligned} \quad (7.92)$$

where $R^{ab}(h)$ is defined by $hI^a h^{-1} = R^{ab} h^b$ and k does not contribute. A^a is the h -gauge transform of a^a , which depends on six real parameters (w^a). The three gauge parameters of $h \in O(3)$, along with the six, give the nine functions needed to parametrize a general $O(3)$ - (or $SU(2)$ -)potential in three dimensions. The Chern–Simons form is

$$\begin{aligned} \text{CS}(A) &= \frac{1}{16\pi^2} \left(A^a dA^a + \frac{1}{3} \epsilon_{abc} A^a A^b A^c \right) \\ &= \frac{1}{16\pi^2} \left(a^a da^a + \frac{1}{3} \epsilon_{abc} a^a a^b a^c \right) - d \left[\frac{1}{16\pi^2} (dh h^{-1})^a a^a \right] + \frac{1}{24\pi^2} \text{Tr}(h^{-1} dh)^3. \end{aligned} \tag{7.93}$$

The second equality reflects the usual response of the Chern–Simons density to gauge transformations. Using the explicit form of a^a as given in (7.92), we can further reduce this. Indeed we find

$$\begin{aligned} a^a da^a + \frac{1}{3} \epsilon_{abc} a^a a^b a^c &= (-2) \frac{(\bar{\mathbf{w}} \times d\bar{\mathbf{w}}) \cdot \rho + (\mathbf{w} \times d\mathbf{w}) \cdot \bar{\rho}}{\left[1 + \mathbf{w} \cdot \bar{\mathbf{w}} - \frac{1}{4} (\mathbf{w} \times \bar{\mathbf{w}})^2 \right]^2}, \\ \rho_k &\equiv \frac{1}{2} \epsilon_{ijk} d\bar{w}^i d\bar{w}^j. \end{aligned} \tag{7.94}$$

Upon defining an Abelian potential

$$a = \frac{\mathbf{w} \cdot d\bar{\mathbf{w}} - \bar{\mathbf{w}} \cdot d\mathbf{w}}{1 + \mathbf{w} \cdot \bar{\mathbf{w}} - \frac{1}{4} (\mathbf{w} \times \bar{\mathbf{w}})^2}, \tag{7.95}$$

we can easily check that $a da$ reproduces (7.94):

$$\text{CS}(A) = \frac{1}{16\pi^2} a da + d \left[\frac{(dh h^{-1})^a a^a}{16\pi^2} \right] + \frac{1}{48\pi^2} \text{Tr}(h^{-1} dh)^3. \tag{7.96}$$

If desired, the Abelian potential a can now be written in the Clebsch form making $a da$ into a total derivative, while the remaining two terms already are total derivatives, though in a ‘hidden’ form for the last expression. This completes our construction.

7.4.2. Total derivative form for the Chern–Simons term on a Kähler manifold. In four-dimensional space (x_1, x_2, x_3, x_4) we can introduce complex coordinates (holomorphic and antiholomorphic) appropriate to Kähler (even-dimensional) manifold:

$$(z, \bar{z}) = \frac{1}{\sqrt{2}}(x_1 \pm ix_2), \quad (w, \bar{w}) = \frac{1}{\sqrt{2}}(x_3 \pm ix_4). \tag{7.97}$$

It is then required that the holomorphic and antiholomorphic components of the curvature $F_{\mu\nu}$ vanish [63]:

$$F_{zw} = F_{\bar{z}\bar{w}} = 0. \tag{7.98}$$

It follows that

$$\begin{aligned} A_z &= U^{-1} \partial_z U, & A_w &= U^{-1} \partial_w U, \\ A_{\bar{z}} &= -(\partial_{\bar{z}} U^\dagger)(U^{-1})^\dagger, & A_{\bar{w}} &= -(\partial_{\bar{w}} U^\dagger)(U^{-1})^\dagger, \end{aligned} \tag{7.99}$$

and after a further complex gauge transformation we may replace (7.95) by

$$A_z = g^{-1} \partial_z g \quad A_w = g^{-1} \partial_w g \quad A_{\bar{z}} = 0 = A_{\bar{w}} \tag{7.100}$$

with Hermitian $g = U^\dagger U$. When we define ∂_\pm as the holomorphic and antiholomorphic derivatives $d_+ = dz \frac{\partial}{\partial z} + dw \frac{\partial}{\partial w}$, $d_- = d\bar{z} \frac{\partial}{\partial \bar{z}} + d\bar{w} \frac{\partial}{\partial \bar{w}}$, then $A = A_z dz + A_w dw + A_{\bar{z}} d\bar{z} +$

$A_{\bar{w}} d\bar{w} = g^{-1} d_+ g \equiv a$ (with requirement (7.98) and in the selected gauge (7.100)). This has the property of Abelianizing the gauge expressions:

$$F = d_- a, \quad \text{Tr } F^2 = \text{Tr}(d_- a da) = d_- \text{Tr}(a d_- a). \quad (7.101)$$

The last formula is also consistent with the Chern–Simons formula, since the cubic contribution in (7.54) vanishes, leaving the Abelianized quantity

$$\text{CS} = \text{Tr}(a d_- a). \quad (7.102)$$

Additionally, one can show that

$$\text{CS} = \text{Tr}(a d_- a) = d_+ \Omega + d_- \Phi. \quad (7.103)$$

(In fact the above is established by varying $\text{Tr}(a d_- a)$.) In this way one arrives at the final result [63]

$$\text{Tr } F^2 = d_- d_+ \Omega. \quad (7.104)$$

It is to be emphasized that (7.103) holds only with the holomorphic restriction (7.98), and gauge choice (7.100). No explicit formula for Ω is available; it is determined by ‘intergrating’ a variation:

$$\delta \Omega = 2 \text{Tr } a d_- v \quad v \equiv g^{-1} \delta g. \quad (7.105)$$

For Φ , which however does not contribute to $\text{Tr } F^2$, but is needed for a reconstruction of the Chern–Simons term (7.103), we find

$$\delta \Phi = \text{Tr}(a d_+ v). \quad (7.106)$$

Evidently, this construction does not give a parametrization for an arbitrary vector potential, only one which can be gauged to (7.3.28).

It is instructive to see some explicit expressions. In the Abelian case, with $A = a = d_+ \theta$, we have $F = d_- d_+ \theta$, $F^2 = (d_- d_+ \theta)(d_- d_+ \theta) = d_- (d_+ \theta d_- d_+ \theta)$, $\text{CS} = AF = d_+ \theta d_- d_+ \theta = d_+ (\theta d_- d_+ \theta)$, $\Omega = \theta d_- d_+ \theta$, $\Phi = 0$. Note that the Abelian connection is parametrized in terms of one function θ , or two functions if the gauge freedom is included. But a general connection in 4-space, requires four functions.

In the non-Abelian $SU(2)$ case, we take $g = \exp(\frac{\sigma^a}{2} \theta^a)$, then

$$A \equiv a = g^{-1} d_+ g = (d_+ (\hat{\theta}^a \sinh \theta) + (\cosh \theta - 1)(i \varepsilon_{abc} d_+ \hat{\theta}^b \hat{\theta}^c - \hat{\theta}^a d_+ \theta)) \frac{\sigma^a}{2} \quad (7.107)$$

with $\theta \equiv \sqrt{\theta^a \theta^a}$, $\hat{\theta}^a \equiv \theta^a / \theta$. The Chern–Simons term is reconstructed from

$$\Omega = \frac{1}{2} d_+ \theta d_- \theta + (\cosh \theta - 1) d_+ \hat{\theta}^a d_- \hat{\theta}^a + i(\sinh \theta - \theta) \varepsilon_{abc} \hat{\theta}^a d_+ \hat{\theta}^b d_- \hat{\theta}^c \quad (7.108a)$$

$$\Phi = -\frac{i}{2} (\sinh \theta - \theta) \varepsilon_{abc} \hat{\theta}^a d_+ \hat{\theta}^b d_- \hat{\theta}^c. \quad (7.108b)$$

Note that three functions are involved (θ^a) in the parametrization of the connection; six if the gauge freedom is included. However an $SU(2)$ connection on a 4-space requires 12 functions.

Appendix A. Canonical formalism determined by a Lagrangian

A.1. Easy case

We begin with a Lagrangian that is first order in time. This entails no loss of generality because all second-order Lagrangians can be converted to first order by the familiar Legendre transformation that produces a Hamiltonian: $H(p, q) = p\dot{q} - L(\dot{q}, q)$, where $p \equiv \partial L / \partial \dot{q}$.

The equations of motion obtained by taking the Euler–Lagrange derivative with respect to p and q of the Lagrangian $L(\dot{p}, p; \dot{q}, q) \equiv p\dot{q} - H(p, q)$ coincide with the ‘usual’ equations of motion obtained by taking the q Euler–Lagrange derivative of $L(\dot{q}, q)$. (In fact $L(\dot{p}, p; \dot{q}, q)$ does not depend on \dot{p} .) Moreover, some Lagrangians possess only a first-order formulation (for example, Lagrangians for Schrödinger or Dirac fields; also the Klein–Gordon Lagrangian in light-cone coordinates is first order in the light-cone ‘time’ derivative).

Denoting all variables by the generic symbol ξ^i , the most general first-order Lagrangian is

$$L = a_i(\xi)\dot{\xi}^i - H(\xi). \tag{A.1}$$

Note that although we shall ultimately be interested in fields defined on spacetime, for the present didactic purpose it suffices to consider variables $\xi^i(t)$ that are functions only of time. The Euler–Lagrange equation that is implied by (A.1) reads

$$f_{ij}(\xi)\dot{\xi}^j = \frac{\partial H(\xi)}{\partial \xi^i} \tag{A.2}$$

where

$$f_{ij}(\xi) = \frac{\partial a_j(\xi)}{\partial \xi^i} - \frac{\partial a_i(\xi)}{\partial \xi^j}. \tag{A.3}$$

The first term in (A.1) determines the canonical 1-form, $a_i(\xi)\dot{\xi}^i dt = a_i(\xi) d\xi^i$, while f_{ij} gives the symplectic 2-form, $da_i(\xi) d\xi^i = \frac{1}{2}f_{ij}(\xi) d\xi^i d\xi^j$.

To set up a canonical formalism, we proceed directly. We *do not* make the frequently heard statement that ‘the canonical momenta $\partial L/\partial \dot{\xi}^i = a_i(\xi)$ are constrained to depend on the coordinates ξ^j ’, and we *do not* embark on Dirac’s method for constrained systems [7].

In fact, if the matrix f_{ij} possesses the inverse f^{ij} there are no constraints. Then (A.2) implies

$$\dot{\xi}^i = f^{ij}(\xi) \frac{\partial H(\xi)}{\partial \xi^j}. \tag{A.4}$$

When one wants to express this equation of motion by bracketing with the Hamiltonian

$$\dot{\xi}^i = \{H(\xi), \xi^i\} = \{\xi^j, \xi^i\} \frac{\partial H(\xi)}{\partial \xi^j}, \tag{A.5}$$

one is led to postulating the fundamental bracket as

$$\{\xi^i, \xi^j\} = -f^{ij}(\xi). \tag{A.6}$$

The bracket between functions of ξ is then defined by

$$\{F_1(\xi), F_2(\xi)\} = -\frac{\partial F_1(\xi)}{\partial \xi^i} f^{ij} \frac{\partial F_2(\xi)}{\partial \xi^j}. \tag{A.7}$$

One verifies that (A.6) and (A.7) satisfy the Jacobi identity by virtue of the Bianchi identity when f is given by (A.3):

$$\frac{\partial}{\partial \xi^i} f_{jk} + \frac{\partial}{\partial \xi^j} f_{ki} + \frac{\partial}{\partial \xi^k} f_{ij} = 0. \tag{A.8}$$

A.2. Difficult case

When f_{ij} is singular, we may still proceed in the following manner [8]. Let us suppose that f_{ij} possesses N zero modes $p_{(n)}^i$,

$$p_{(n)}^i f_{ij} = 0 \quad n = 1, \dots, N. \quad (\text{A.9})$$

If we use a rank N projection operator P_i^j that satisfies

$$P_i^j P_j^k = P_i^k, \quad P_i^j f_{jk} = 0, \quad (\text{A.10})$$

it is possible to find an inverse for f_{ij} on the projected subspace. Namely, the ‘inverse’ f^{ij} is uniquely determined by

$$\begin{aligned} f_{ik} f^{kj} &= \delta_i^j - P_i^j \\ f^{kj} &= -f^{jk}, \quad f^{ik} P_k^j = 0. \end{aligned} \quad (\text{A.11})$$

Once f^{ij} is constructed, we define the Poisson bracket for functions of ξ^i by (A.7).

It still remains to verify the Jacobi identity. An easy computation shows that (A.7) continues to satisfy that identity provided

$$P_i^j \frac{\delta}{\delta \xi^j} F_\ell = 0. \quad (\text{A.12})$$

Hence we use the brackets (A.7) only between functions $F_\ell(\xi)$ that satisfy the admissibility criterion (A.12).

A.3. Obstructions to a canonical formalism

Our problem in connection with Eulerian fluid mechanics is in fact the inverse of what has been summarized above. From (1.39), (1.42) and (1.43), we know the form of f^{ij} and that the Jacobi identity holds. We then wish to determine the inverse f_{ij} , and then a_i from (A.3). Since we know the Hamiltonian from (1.36), construction of the Lagrangian (A.1) should follow immediately.

However, an obstacle arises: since there exist Casimir invariants $C(\xi)$ whose brackets with all the ξ^i vanish, then

$$0 = \{\xi^i, C(\xi)\} = -f^{ij} \frac{\partial}{\partial \xi^j} C(\xi). \quad (\text{A.13})$$

That is, f^{ij} has zero modes $\frac{\partial}{\partial \xi^j} C(\xi)$, and the inverse of f^{ij} , namely the symplectic 2-form f_{ij} , does not exist. In that case, something has to be done to neutralize the Casimirs.

A.4. Canonical transformations

It is interesting, and will be useful, to develop the theory further into a discussion of canonical transformations [9]. A canonical transformation is a transformation on the phase space coordinates ξ^i , given infinitesimally by a vector function,

$$\delta \xi^i = -v^i(\xi), \quad (\text{A.14})$$

which leaves the symplectic 2-form f_{ij} invariant:

$$\delta f_{ij} = v^m \partial_m f_{ij} + \partial_i v^m f_{mj} + \partial_j v^m f_{im} = 0. \quad (\text{A.15a})$$

(The expression in (A.15a) is the Lie derivative of f_{ij} with respect to the vector field v^m .) Use of the Bianchi identity (A.8) allows casting (A.15a) into

$$\partial_i (v^m f_{mj}) - \partial_j (v^m f_{mi}) = 0. \quad (\text{A.15b})$$

This shows that the quantity $v^m f_{mi}$ may be presented as

$$v^m f_{mi} = \frac{\partial G(\xi)}{\partial \xi^i}. \tag{A.16}$$

$G(\xi)$ is called a ‘generator’ for the vector field v^m . Conversely, when f_{mi} possesses an inverse, given any function $G(\xi)$ on phase space we can define a vector field v^m . Equation (A.15b) is the general requirement for v^i to define a canonical transformation. Equation (A.16) is a necessary and sufficient condition, locally on phase space. If the phase space has non-trivial topology, one can have more general solutions to the condition (A.15).

The change of a function $F(\xi)$ on phase space, due to a canonical transformation, is given by

$$\begin{aligned} \delta F &= -\frac{\partial F}{\partial \xi^m} v^m(\xi) \\ &= -\frac{\partial G}{\partial \xi^i} f^{im} \frac{\partial F}{\partial \xi^m} = \{G, F\} \end{aligned} \tag{A.17}$$

where (A.6), (A.7) and (A.16) have been used. This shows that indeed G generates the infinitesimal canonical transformation by Poisson bracketing.

Note that the bracket (A.17) has been established without the inverse 2-form f^{ij} . This enjoys an advantage over explicit evaluations relying on the methods presented above in section A.1, and especially in section A.2, where a projected inverse is employed. Moreover, we see that the admissibility criterion (A.12) is automatically satisfied by any generator G that solves (A.16). This follows immediately from (A.9).

Appendix B. Clebsch parametrization and the Chern–Simons term

We elaborate on the Clebsch parametrization for a vector field [11], which was presented for the velocity vector in (1.68). Here we shall use the notation of electromagnetism and discuss the Clebsch parametrization of a vector potential \mathbf{A} , which also leads to the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. (Of course the same observations apply when the vector in question is the velocity field \mathbf{v} , with $\nabla \times \mathbf{v}$ giving the vorticity.)

The familiar parametrization of a 3-component vector employs a scalar function θ (the ‘gauge’ or ‘longitudinal’ part) and a 2-component transverse vector \mathbf{A}_T : $\mathbf{A} = \nabla\theta + \mathbf{A}_T$, $\nabla \cdot \mathbf{A}_T = 0$. This decomposition is unique and invertible (on a space with simple topology). In contrast, the Clebsch parametrization uses three scalar functions, θ , α and β ,

$$\mathbf{A} = \nabla\theta + \alpha \nabla\beta, \tag{B.1}$$

which are not uniquely determined by \mathbf{A} (see below). The associated magnetic field reads

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla\alpha \times \nabla\beta. \tag{B.2}$$

Repeating the above in form notation, the 1-form $A = A_i dr^i$ is presented as

$$A = d\theta + \alpha d\beta, \tag{B.3}$$

and the 2-form is

$$dA = d\alpha d\beta. \tag{B.4}$$

Darboux’s theorem¹⁴ ensures that the Clebsch parametrization is attainable locally in space (in the form (B.3)). Additionally, an explicit construction of α , β and θ can be given by the following procedure [13].

¹⁴ A constructive discussion of the Darboux theorem can be found in the second work of [7].

Solve the equations

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}, \quad (\text{B.5a})$$

which may also be presented as

$$\varepsilon^{ijk} dr^j B^k = 0. \quad (\text{B.5b})$$

Solutions of these relations define two surfaces, called ‘magnetic surfaces’, that are given by equations of the form

$$S_n(\mathbf{r}) = \text{constant} \quad (n = 1, 2). \quad (\text{B.6})$$

It follows from (B.5) that these also satisfy

$$\mathbf{B} \cdot \nabla S_n = 0 \quad (n = 1, 2), \quad (\text{B.7})$$

that is, the normals to S_n are orthogonal to \mathbf{B} , or \mathbf{B} is parallel to the tangent of S_n . The intersection of the two surfaces forms the so-called ‘magnetic lines’, that is, loci that solve the dynamical system

$$\frac{d\mathbf{r}(\tau)}{d\tau} = \mathbf{B}(\mathbf{r}(\tau)), \quad (\text{B.8})$$

where τ is an evolution parameter. Finally, the remaining Monge potentials α and β are constructed as functions of \mathbf{r} only through a dependence on the magnetic surfaces,

$$\alpha(\mathbf{r}) = \alpha(S_n(\mathbf{r})), \quad \beta(\mathbf{r}) = \beta(S_n(\mathbf{r})), \quad (\text{B.9})$$

so that

$$\nabla\alpha \times \nabla\beta = (\nabla S_1 \times \nabla S_2) \varepsilon^{mn} \frac{\partial\alpha}{\partial S_m} \frac{\partial\beta}{\partial S_n}. \quad (\text{B.10})$$

Evidently as a consequence of (B.7), $\nabla\alpha \times \nabla\beta$ in (B.10) is parallel to \mathbf{B} , and because \mathbf{B} is divergence-free α and β can be adjusted so that the norm of $\nabla\alpha \times \nabla\beta$ coincides with $|\mathbf{B}|$. Once α and β have been fixed in this way, θ can easily be computed from $\mathbf{A} - \alpha\nabla\beta$.

Neither the individual magnetic surfaces nor the Monge potentials are unique. (By viewing A as a canonical 1-form, it is clear that expression (B.3) retains its form after a canonical transformation of α, β .) One may therefore require that the Monge potentials α and β simply coincide with the two magnetic surfaces: $\alpha = S_1, \beta = S_2$. Nevertheless, for a given \mathbf{A} and \mathbf{B} it may not be possible to solve (B.5) explicitly.

The Chern–Simons integrand $\mathbf{A} \cdot \mathbf{B}$ becomes in the Clebsch parametrization

$$\mathbf{A} \cdot \mathbf{B} = \nabla\theta \cdot (\nabla\alpha \times \nabla\beta) = \nabla \cdot (\theta\mathbf{B}) = \mathbf{B} \cdot \nabla\theta. \quad (\text{B.11})$$

Thus having identified the Monge potentials α and β with the two magnetic surfaces, we deduce from (B.7) and (B.11) three equations for the three functions (θ, α, β) that comprise the Clebsch parametrization:

$$\mathbf{B} \cdot \nabla\alpha = \mathbf{B} \cdot \nabla\beta = 0 \quad \mathbf{B} \cdot \nabla\theta = \text{Chern–Simons density } \mathbf{A} \cdot \mathbf{B}. \quad (\text{B.12})$$

Equation (B.11) also shows that in the Clebsch parametrization the Chern–Simons density becomes a total derivative:

$$\mathbf{A} \cdot \mathbf{B} = \nabla \cdot (\theta\mathbf{B}). \quad (\text{B.13})$$

This does *not* mean that the Clebsch parametrization is unavailable when the Chern–Simons integral over all space is nonzero. Rather for a non-vanishing integral and well-behaved \mathbf{B} field, one must conclude that the Clebsch function θ is singular either in the finite volume of the integration region or on the surface at infinity bounding the integration domain. Then

the Chern–Simons volume integral over (Ω) becomes a surface integral on the surfaces $(\partial\Omega)$ bounding the singularities:

$$\int_{\Omega} d^3r \mathbf{A} \cdot \mathbf{B} = \int_{\partial\Omega} d\mathbf{S} \cdot (\theta\mathbf{B}). \quad (\text{B.14})$$

Equation (B.14) shows that the Chern–Simons integral measures the magnetic flux, modulated by θ and passing through the surfaces that surround the singularities of θ .

The following explicit example illustrates the above points.

Consider the vector potential whose spherical components are given by

$$A_r = (\cos \Theta)a'(r), \quad A_{\Theta} = -(\sin \Theta)\frac{1}{r} \sin a(r), \quad A_{\Phi} = -(\sin \Theta)\frac{1}{r}(1 - \cos a(r)). \quad (\text{B.15})$$

(r and Θ , Φ denote the conventional radial coordinate and the polar, azimuthal angles.) The function $a(r)$ is taken to vanish at the origin, and to behave as $2\pi\nu$ at infinity (ν integer or half-integer). The corresponding magnetic field reads

$$\begin{aligned} B_r &= -2(\cos \Theta)\frac{1}{r^2}(1 - \cos a(r)), \\ B_{\Theta} &= (\sin \Theta)\frac{1}{r}a'(r) \sin a(r), \\ B_{\Phi} &= (\sin \Theta)\frac{1}{r}a'(r)(1 - \cos a(r)), \end{aligned} \quad (\text{B.16})$$

and the Chern–Simons integral—also called the ‘magnetic helicity’ in the electro-dynamical context—is quantized (in multiples of $16\pi^2$) by the behaviour of $a(r)$ at infinity:

$$\begin{aligned} \int d^3r \mathbf{A} \cdot \mathbf{B} &= -8\pi \int_0^{\infty} dr \frac{d}{dr}(a(r) - \sin a(r)) \\ &= -16\pi^2\nu. \end{aligned} \quad (\text{B.17})$$

In spite of the non-vanishing magnetic helicity, a Clebsch parametrization for (B.15) is readily constructed. In form notation, it reads

$$A = d(-2\Phi) + 2 \left(1 - \left(\sin^2 \frac{a}{2}\right) \sin^2 \Theta\right) d \left(\Phi + \tan^{-1} \left[\left(\tan \frac{a}{2}\right) \cos \Theta\right]\right). \quad (\text{B.18})$$

The magnetic surfaces can be taken from formula (B.18) to coincide with the Monge potentials:

$$S_1 = 2 \left(1 - \left(\sin^2 \frac{a}{2}\right) \sin^2 \Theta\right) = \text{constant} \quad S_2 = \Phi + \tan^{-1} \left[\left(\tan \frac{a}{2}\right) \cos \Theta\right] = \text{constant}. \quad (\text{B.19})$$

The magnetic lines are determined by the intersection of S_1 and S_2 ,

$$\cos \frac{a}{2} = \varepsilon \cos(\Phi - \phi_0) \quad \sin \Theta = \sqrt{\frac{1 - \varepsilon^2}{1 - \varepsilon^2 \cos^2(\Phi - \phi_0)}} \quad (\text{B.20})$$

where ε and ϕ_0 are constants. The potential $\theta = -2\Phi$ is multivalued. Consequently the ‘surface’ integral determining the Chern–Simons term reads

$$\int d^3r \mathbf{A} \cdot \mathbf{B} = \int d^3r \nabla \cdot (-2\Phi\mathbf{B}) = -4\pi \int_0^{\infty} r dr \int_0^{\pi} d\Theta B_{\Phi}|_{\Phi=2\pi}. \quad (\text{B.21})$$

That is, the magnetic helicity is the flux of the toroidal magnetic field through the positive- x (x, z) half-plane.

Finally, we remark on a subtle property of the Clebsch decomposition when used in variational calculations [14]. Consider an ‘action’ of the form

$$I = I_0(\mathbf{B}) + \frac{\mu}{2} \int_{\Omega} d^3r \mathbf{A} \cdot \mathbf{B}. \quad (\text{B.22})$$

Variation of \mathbf{A} gives

$$\delta I = \int_{\Omega} d^3r (\nabla \times \mathcal{B} + \mu \mathbf{B}) \cdot \delta \mathbf{A}, \quad (\text{B.23})$$

where $\mathcal{B}(\mathbf{r}) \equiv \frac{\delta I_0(\mathbf{B})}{\delta \mathbf{B}(\mathbf{r})}$. Demanding that I be stationary against variations of \mathbf{A} requires the vanishing of the term in parentheses, which is transverse, since the transverse part of the variation $\delta \mathbf{A}$ is arbitrary:

$$\nabla \times \mathcal{B} + \mu \mathbf{B} = 0. \quad (\text{B.24})$$

Now let us examine the same problem in the Clebsch parametrization. The Chern–Simons contribution to (B.22) reads, according to (B.11),

$$\frac{\mu}{2} \int_{\Omega} d^3r \nabla \cdot (\theta \mathbf{B}) = \frac{\mu}{2} \int_{\partial \Omega} d\mathbf{S} \cdot \theta \mathbf{B}. \quad (\text{B.25})$$

In the gauge $\theta = 0$ (B.25) vanishes, and in any gauge it has no bulk contribution, so its variation will never produce the second left-hand term in (B.24). So how is (B.24) regained?

Returning to (B.22), we accept the fact that the variation of the last term vanishes, while the variation of the first leaves

$$\delta I = \int_{\Omega} d^3r (\nabla \times \mathcal{B}) \cdot (\nabla \beta \delta \alpha - \nabla \alpha \delta \beta). \quad (\text{B.26})$$

Since $\delta \alpha$ and $\delta \beta$ are arbitrary, I is stationary provided

$$(\nabla \times \mathcal{B}) \cdot \nabla \beta = 0, \quad (\text{B.27a})$$

$$(\nabla \times \mathcal{B}) \cdot \nabla \alpha = 0. \quad (\text{B.27b})$$

We obtain two equations, which imply by (B.12)

$$\nabla \times \mathcal{B} + \mu(\mathbf{r}) \mathbf{B} = 0. \quad (\text{B.28})$$

Transversality of $\nabla \times \mathbf{B}$ and \mathbf{B} further requires that $\mathbf{B} \cdot \nabla \mu(\mathbf{r}) = 0$, i.e., a non-vanishing $\nabla \mu(\mathbf{r})$ (= non-constant μ) must lie in the $(\nabla \alpha, \nabla \beta)$ plane.

Since (B.28) is weaker than the parametrization independent (B.24), we conclude that the Clebsch parametrization is somewhat incomplete, when used in variational calculations that ignore surface terms. This is similar to the fact that in the Clebsch parametrization gauge potentials, which carry a non-vanishing Chern–Simons term (= velocities with non-vanishing vortex helicity), encode the non-vanishing value in a surface term.

In section 7.4, we shall present a different method, based on group theory, for obtaining the Clebsch parametrization. This approach is then generalized to non-Abelian vector fields.

Appendix C. The integrability of the cubic potential in $d = 1$

Although it does not belong to the models that we have discussed, the cubic potential for one-dimensional motion, $V(\rho) = \ell \rho^3/3$, is especially interesting because it is secretly free (see (1.57b))—a fact that is exposed when Riemann coordinates are employed. For this problem, these read $R_{\pm} = p \pm \sqrt{2\ell} \rho$ and again they are just the velocity (\pm) the speed of sound. In contrast to (3.19) the Euler and continuity equations for this system decouple: $\dot{R}_{\pm} = -R_{\pm} R'_{\pm}$.

Indeed, it is seen that R_{\pm} satisfy essentially the free Euler equation (compare with (1.84b) at $V'' = 0$ and identify R_{\pm} with v). Consequently, the solution (1.85)–(1.89) works here as well.

Recall the previous remark in section 1.27 on the Schrödinger group (Galileo and $SO(2,1)$): in one dimension the cubic potential is invariant against this group of transformations, and in all dimensions the free theory is invariant. Therefore, a natural speculation is that the secretly non-interacting nature of the cubic potential in one dimension is a consequence of Schrödinger group invariance.

Another interesting fact about a one-dimensional nonrelativistic fluid with cubic potential is that it also arises in a collective, semiclassical description of nonrelativistic free fermions in one dimension, where the cubic potential reproduces fermion repulsion [27]. In spite of the non-linearity of the fluid model's equations of motion, there is no interaction in the underlying fermion dynamics. Thus, the presence of the Schrödinger group and the equivalence to free equations for this fluid system is an understandable consequence.

Appendix D. The charge density algebra

The portion of the Lagrange density (5.28) that determines the Poisson bracket is

$$\mathcal{L}_{\text{canonical}} = \rho 2 \text{Tr} K g^{-1} \dot{g} = \rho 2 \text{Tr} Q \dot{g} g^{-1}. \quad (\text{D.1})$$

where $Q \equiv g K g^{-1}$. The phase space is parametrized by the scalar density ρ and parameters φ_a specifying the elements g of the group G . Thus the phase space is identified as

$$\{\text{set of all maps } \rho(\mathbf{r}) = \mathbf{R}^3 \rightarrow \mathbf{R}_+, g(\mathbf{r}) = \mathbf{R}^3 \rightarrow G\}.$$

With a parametrization of the group element, e.g., $g(\varphi) = e^{T^a \varphi_a}$, one notes that $\dot{g} g^{-1}$ has the form $\dot{\varphi}_a C^a_b(\varphi) T^b$, where the non-singular matrix $C^a_b(\varphi)$ is defined by

$$C^a_b(\varphi) T^b = \frac{\partial g(\varphi)}{\partial \varphi_a} g^{-1}(\varphi). \quad (\text{D.2})$$

Thus

$$\mathcal{L}_{\text{canonical}} = -\rho \dot{\varphi}_a C^a_b Q_b = -\dot{\varphi}_a C^a_b \rho_b. \quad (\text{D.3})$$

We give a new name to the combination $C^a_b \rho_b$,

$$\Pi^a \equiv -C^a_b \rho_b. \quad (\text{D.4})$$

The (D.3) reads

$$\mathcal{L}_{\text{canonical}} = \Pi^a \dot{\varphi}_a. \quad (\text{D.5})$$

Consequently, applying the formalism explained in appendix A.1, we conclude immediately that Π^a and φ_a are canonically conjugate. Moreover, the charge density can be expressed in terms of Π^a and c^b_a , the inverse of C^a_b ,

$$\rho_a = -c^a_b \Pi^b. \quad (\text{D.6})$$

The non-Abelian charge density ρ_a is a function of (t, \mathbf{r}) and for (5.20) we need the bracket with another density evaluated at (t, \mathbf{r}') . (The common t -dependence is suppressed.) Since the dependence of c^a_b on φ involves no spatial derivatives of φ , it is clear that the brackets will be local in $\mathbf{r} - \mathbf{r}'$, just as is the bracket between φ and Π :

$$\begin{aligned} \{\rho_a(\mathbf{r}), \rho_b(\mathbf{r}')\} &= \left(c^a_{a'} \frac{\partial c^b_{b'}}{\partial \varphi_{a'}} \Pi^{b'} - a \leftrightarrow b \right) \delta(\mathbf{r} - \mathbf{r}') \\ &= \left(-c^a_{a'} c^b_{c'} \frac{\partial C^{c'}_{c''}}{\partial \varphi_{a'}} c^{c''}_{b'} \Pi^{b'} - a \leftrightarrow b \right) \delta(\mathbf{r} - \mathbf{r}') \\ &= \left(c^a_{a'} c^b_{c'} \frac{\partial C^{c'}_{c''}}{\partial \varphi_{a'}} \rho_{c''} - a \leftrightarrow b \right) \delta(\mathbf{r} - \mathbf{r}'). \end{aligned} \quad (\text{D.7})$$

To evaluate the derivative with respect to φ , return to (D.2) and observe

$$\begin{aligned}\frac{\partial C^{c'c''}}{\partial \varphi_{a'}} &= -\frac{\partial}{\partial \varphi_{a'}} 2 \operatorname{Tr} \frac{\partial g}{\partial \varphi_{c'}} g^{-1} T^{c''} \\ &= -2 \operatorname{Tr} \left(\frac{\partial^2 g}{\partial \varphi_{a'} \partial \varphi_{c'}} g^{-1} - C^{c'd'} T^{d'} C^{a'd''} T^{d''} \right) T^{c''}.\end{aligned}\quad (\text{D.8})$$

The first term in the parentheses is symmetric in (a', c') ; when inserted in (D.7) it produces a symmetric contribution in (a, b) and does not contribute when antisymmetrization in (a, b) is effected. What is left establishes (5.20):

$$\begin{aligned}\{\rho_a(\mathbf{r}), \rho_b(\mathbf{r}')\} &= (c^a{}_{a'} c^b{}_{c'} C^{c'd'} C^{a'd''} 2 \operatorname{Tr} T^{d'} T^{d''} T^{c''} \rho_{c''} - a \leftrightarrow b) \delta(\mathbf{r} - \mathbf{r}') \\ &= -(2 \operatorname{Tr} T^a T^b T^{c''} \rho_{c''} - a \leftrightarrow b) \delta(\mathbf{r} - \mathbf{r}') \\ &= -2 \operatorname{Tr} f_{abd} T^d T^{c''} \rho_{c''} \delta(\mathbf{r} - \mathbf{r}') \\ &= f_{abc} \rho_c(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}').\end{aligned}\quad (\text{D.9})$$

For simplicity, here we examined the single-channel case. The multi-channel case can be treated similarly.

It is instructive to rederive the above result by using the general theory of canonical transformations, described in appendices A.1 and A.4.

From (D.3) we see that the canonical 1-form has components

$$a_\rho = 0, \quad a_{\varphi_a} = -\rho C^a{}_b Q_b = -C^a{}_b \rho b. \quad (\text{D.10})$$

(These generalize a_i in (A.1).) The symplectic 2-form (generalizing f_{ij} in (A.3)) reads

$$\begin{aligned}f_{\rho\rho}(\mathbf{r}, \mathbf{r}') &= 0, \\ f_{\rho\varphi_a}(\mathbf{r}, \mathbf{r}') &= -\delta(\mathbf{r} - \mathbf{r}') C^a{}_b Q_b, \\ f_{\varphi_a\varphi_b}(\mathbf{r}, \mathbf{r}') &= \delta(\mathbf{r} - \mathbf{r}') \rho C^a{}_c C^b{}_d Q_e f_{cde}.\end{aligned}\quad (\text{D.11})$$

(The δ -function arises because the 1-form is a function of the coordinates and involves no coordinate derivatives.) It follows from (A.12) that the generator of an infinitesimal canonical transformation

$$\delta\rho = -v^\rho \quad \delta\varphi_a = -v^{\varphi_a} \quad (\text{D.12})$$

obeys

$$\int d\mathbf{r}' [v^{\varphi_a} f_{\varphi_a\rho}] = v^{\varphi_a} C^a{}_b Q_b = \frac{\delta G}{\delta\rho} \quad (\text{D.13a})$$

$$\int d\mathbf{r}' [v^\rho f_{\rho\varphi_a} + v^{\varphi_b} f_{\varphi_b\varphi_a}] = -v^\rho C^a{}_b Q_b - v^{\varphi_b} \rho C^a{}_c C^b{}_d Q_e f_{cde} = \frac{\delta G}{\delta\varphi_a}. \quad (\text{D.13b})$$

Let us now consider the left translation of g : $\delta g = \epsilon_a T^a g$, or equivalently $\delta\varphi_a = \epsilon_b c^b{}_a$, $\delta Q_a = f_{abc} \epsilon_b Q_c$, $\delta\rho_a = f_{abc} \epsilon_b \rho_c$. Here ϵ_a is a function of \mathbf{r} . Thus we have $v^\rho = 0$, $v^{\varphi_a} = \epsilon_b c^b{}_a$ and (D.13) are solved by

$$G = - \int d\mathbf{r} \rho Q_a \epsilon_a = - \int d\mathbf{r} \epsilon_a \rho^a. \quad (\text{D.14})$$

From the general theory (A.13), we learn that by Poisson bracketing G generates the above transformation on any function of the phase space variables. Thus

$$\left\{ \int d\mathbf{r} \epsilon^a(\mathbf{r}) \rho_a(\mathbf{r}), \rho_b(\mathbf{r}') \right\} = -\delta \rho_b(\mathbf{r}') \\ = f_{abc} \epsilon_b(\mathbf{r}') \rho_c(\mathbf{r}'). \quad (\text{D.15})$$

Stripping away the arbitrary function $\epsilon_a(\mathbf{r})$ reproduces (D.9).

Next we consider a right translation of g by the Lie algebra element K , present in the canonical 1-form: $\delta g = gK\lambda$, or equivalently $\delta \varphi_a = \lambda Q_b c^b_a$, leading to $v^\rho = 0$, $v^{\varphi_a} = -\lambda Q_b c^b_a$. Here λ is a function of \mathbf{r} . Equations (D.13) are now solved by

$$G = - \int d\mathbf{r} \lambda \rho, \quad (\text{D.16})$$

when K is normalized to $\text{Tr} K^2 = -\frac{1}{2}$. It then follows from (A.13) that once λ is stripped away,

$$\{\rho(\mathbf{r}), g(\mathbf{r}')\} = -\delta(\mathbf{r} - \mathbf{r}') g K, \quad (\text{D.17})$$

and

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = 0. \quad (\text{D.18})$$

Appendix E. Varying the group element g

We determine the variation of

$$I_0 = \int dt d\mathbf{r} \sum_{s=1}^r j_{(s)}^\mu 2 \text{Tr} K_{(s)} g^{-1} D_\mu g, \quad (\text{E.1})$$

when g is varied either arbitrarily or in the specific manner

$$g^{-1} \delta g = K_{(s')} \lambda, \quad (\text{E.2})$$

where λ is an arbitrary function on spacetime. This will provide the needed results (5.32) and (5.33) for the single-channel situation, as well as for many channels in (5.49) and (5.50), below.

Recall the definitions $Q_{(s)} = g K_{(s)} g^{-1}$ and $D_\mu g = \partial_\mu g + A_\mu g$, which imply $D_\mu g^{-1} = \partial_\mu g^{-1} - g^{-1} A_\mu$. First, the variation of $g^{-1} D_\mu g$ is established:

$$\delta(g^{-1} D_\mu g) = -g^{-1} \delta g g^{-1} D_\mu g + g^{-1} D_\mu \delta g. \quad (\text{E.3})$$

To evaluate the last term, note that $D_\mu \delta g = D_\mu (g g^{-1} \delta g) = (D_\mu g) g^{-1} \delta g + g D_\mu (g^{-1} \delta g)$. Thus

$$\delta(g^{-1} D_\mu g) = \partial_\mu (g^{-1} \delta g) + [g^{-1} D_\mu g, g^{-1} \delta g]. \quad (\text{E.4})$$

Inserting (E.4) into the variation of I_0 in (E.1), integrating by parts, and rearranging the trace with $K_{(s)}$ gives

$$\delta I_0 = - \int dt d\mathbf{r} \sum_{s=1}^r (\partial_\mu j_{(s)}^\mu 2 \text{Tr} K_{(s)} g^{-1} \delta g + j_{(s)}^\mu 2 \text{Tr} [g^{-1} D_\mu g, K_{(s)}] g^{-1} \delta g). \quad (\text{E.5})$$

Considering first arbitrary variations: the vanishing of δI_0 requires

$$\sum_{s=1}^r (\partial_\mu j_{(s)}^\mu K_{(s)} + j_{(s)}^\mu [g^{-1} D_\mu g, K_{(s)}]) = 0 \quad (\text{E.6})$$

or, after sandwiching the above between $g \dots g^{-1}$,

$$\sum_{s=1}^r (\partial_\mu j_{(s)}^\mu Q_{(s)} + j_{(s)}^\mu [D_\mu g g^{-1}, Q_{(s)}]) = 0. \quad (\text{E.7})$$

Finally, we verify that

$$[D_\mu g g^{-1}, Q_{(s)}] = D_\mu Q_{(s)}, \quad (\text{E.8})$$

so that the desired results (5.32) and (5.50) follow:

$$\sum_{s=1}^r (\partial_\mu j_{(s)}^\mu Q_{(s)} + j_{(s)}^\mu D_\mu Q_{(s)}) = D_\mu \left(\sum_{s=1}^r j_{(s)}^\mu Q_{(s)} \right) = D_\mu J^\mu = 0. \quad (\text{E.9})$$

Next we consider the specific variation (E.2) and separate the sum (E.5) into the term $s = s'$ and $s \neq s'$. After a rearrangement of the last term in (E.5), we get

$$\begin{aligned} \delta I_0 = & - \int dt dr (\partial_\mu j_{(s')}^\mu 2 \text{Tr} K_{(s')} K_{(s')} \lambda + j_{(s')}^\mu 2 \text{Tr} g^{-1} D_\mu g [K_{(s')}, K_{(s')}] \lambda) \\ & + \sum_{s \neq s'} (\partial_\mu j_{(s)}^\mu 2 \text{Tr} K_{(s)} K_{(s')} \lambda + j_{(s)}^\mu 2 \text{Tr} g^{-1} D_\mu g [K_{(s)}, K_{(s')}] \lambda). \end{aligned} \quad (\text{E.10})$$

The first commutator vanishes; so does the second when $K_{(s)}$ and $K_{(s')}$ commute, i.e. when they belong to the Cartan subalgebra. Also $2 \text{Tr} K_{(s)} K_{(s')} = -K_{(s)}^a K_{(s')}^a$; for $s' = s$ this is constant, while for $s' \neq s$ it vanishes when it is arranged that distinct elements of the Cartan algebra are selected. Thus for stationary variations $j_{(s)}^\mu$ must be conserved, and (5.33) as well as (5.49) are validated.

Appendix F. Manipulating equation (5.36)

Observe that the first term in (5.36) equals

$$\partial_\mu 2 \text{Tr} Q (D_\nu g) g^{-1} = 2 \text{Tr} ((D_\mu Q) (D_\nu g) g^{-1} + Q (D_\mu D_\nu g) g^{-1} - Q (D_\nu g) g^{-1} (D_\mu g) g^{-1}). \quad (\text{F.1})$$

The first term on the right-hand side is rewritten with the help of (E.8) and combined with the last term, leaving

$$2 \text{Tr} (Q (D_\mu D_\nu g) g^{-1} - Q (D_\mu g) g^{-1} (D_\nu g) g^{-1}).$$

After antisymmetrization in (μ, ν) , the left-hand side of (F.1) reads

$$\begin{aligned} 2 \text{Tr} Q ([D_\mu, D_\nu] g) g^{-1} - [(D_\mu g) g^{-1}, (D_\nu g) g^{-1}] \\ = 2 \text{Tr} (Q F_{\mu\nu} - [Q, (D_\mu g) g^{-1}] (D_\nu g) g^{-1}). \end{aligned} \quad (\text{F.2})$$

When (E.8) is used again, (F.2) becomes the left-hand side of (5.37).

Appendix G. Field-based fluid mechanics

In the body of our review, we presented Eulerian variables, their connection to Lagrangian variables and their equations of motion in a picture derived from an underlying physical reality composed of point particles, whose discrete distribution is approximated by a continuum. For the Abelian case this was developed in section 1.1. The hallmark of this approach is that the fluid current factorizes in an Eckart form. For the Abelian current we have

$$j^\mu = n u^\mu, \quad (\text{G.1})$$

which is no restriction at all, but for the non-Abelian current the further factorization is non-trivial,

$$J_a^\mu = Q_a j^\mu = Q_a n u^\mu. \quad (\text{G.2})$$

Here we present an alternative view of fluid mechanics, which is field based as opposed to particle based. In the Abelian case, it results in fluid equations that coincide with those of the particle-based derivation. This is in keeping with the fact that the Abelian Eckart form is not restrictive. However for the non-Abelian situation, the field-based picture results in equations which are different and much less elegant than the particle-based ones discussed in the body of the text.

A field-based realization of the Euler equations for an Abelian fluid is provided by the Madelung ‘hydrodynamical’ rewriting of the Schrödinger equation [42]:

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\nabla^2\psi. \quad (\text{G.3})$$

(We consider only the free equation.) Upon presenting the wavefunction as

$$\psi(t, \mathbf{r}) = \sqrt{\rho(t, \mathbf{r})} e^{im\theta(t, \mathbf{r})/\hbar} \quad (\text{G.4})$$

we find that the imaginary part of (G.3) results in the continuity equation for $j^\mu = (c\rho, \mathbf{j})$, where ρ is the spatial density $\psi^*\psi$ and the spatial current \mathbf{j} is also the quantum current $(\hbar/m)\text{Im}\psi^*\nabla\psi$. When \mathbf{j} is written as $\mathbf{v}\rho$, \mathbf{v} is identified as $\nabla\theta$; the velocity is irrotational. The real part of (G.3) gives the Bernoulli equation, with a quantum force derived from

$$V = \frac{\hbar^2}{2m^2}(\nabla\sqrt{\rho})^2 = \frac{\hbar^2}{8m^2}\frac{(\nabla\rho)^2}{\rho}. \quad (\text{G.5})$$

The Euler equation follows by taking the gradient of the Bernoulli equation. In this way, we arrive again at the conventional irrotational fluid.

The story changes if we start from a non-Abelian Schrödinger equation. Again we consider the free case with nonrelativistic kinematics. Thus the equation involves a multi-component wavefunction Ψ , with

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m}\nabla^2\Psi. \quad (\text{G.6})$$

The colour degrees of freedom lead to the conserved non-Abelian current:

$$J_a^\mu = (c\rho_a, \mathbf{J}_a), \quad \rho_a = i\Psi^\dagger T^a \Psi, \quad \mathbf{J}_a = \frac{\hbar}{m}\text{Re}\Psi^\dagger T^a \nabla\Psi. \quad (\text{G.7})$$

The singlet current $j^\mu = (c\rho, \mathbf{j}) = (c\Psi^*\Psi, (\frac{\hbar}{m})\text{Im}\Psi^*\nabla\Psi)$ is also conserved.

For definiteness and simplicity, we shall henceforth assume that the group is $SU(2)$ and that the representation is the fundamental one: $T^a = \sigma^a/2i$, $\{T^a, T^b\} = -\delta_{ab}/2$. We shall also set the mass m and Planck’s constant \hbar to unity [43]. The non-Abelian analogue of the Madelung decomposition (G.4) is

$$\Psi = \sqrt{\rho}gu, \quad (\text{G.8})$$

where ρ is the scalar $\Psi^\dagger\Psi$, g is a group element, and u is a constant vector that points in a fixed direction (e.g., for $SU(2)$ the 2-component spinor u could be taken as $u_1 = 1, u_2 = 0$, then $iu^\dagger T^a u = \delta_3^a/2$). The singlet density is ρ , while the singlet current $\mathbf{j} = \text{Im}\Psi^*\nabla\Psi$ is

$$\mathbf{j} = \mathbf{v}\rho, \quad \mathbf{v} \equiv -iu^\dagger g^{-1}\nabla g u. \quad (\text{G.9})$$

With the decomposition (G.8), the colour density (G.7) becomes

$$\rho_a = Q_a\rho, \quad Q_a = iu^\dagger g^{-1}T^a g u = iR_{ab}u^\dagger T^b u = R_{ab}t^b/2, \quad (\text{G.10})$$

where R_{ab} is in the adjoint representation of the group and the unit vector t^a is defined as $t^a/2 = iu^\dagger T^a u$. On the other hand, the colour current reads

$$\mathbf{J}_a = \frac{1}{2}\rho R_{ab}u^\dagger (T^b g^{-1}\nabla g + g^{-1}\nabla g T^b)u, \quad (\text{G.11})$$

which with the introduction of

$$g^{-1}\nabla g \equiv -2\mathbf{v}^a T^a, \quad (\text{G.12})$$

$$\mathbf{v} = \mathbf{v}^a t^a, \quad (\text{G.13})$$

may be presented as

$$\mathbf{J}_a = \frac{\rho}{2} R_{ab} \mathbf{v}^b. \quad (\text{G.14})$$

Unlike the Abelian model, the vorticity is non-vanishing:

$$\nabla \times \mathbf{v}^a = \epsilon^{abc} \mathbf{v}^b \times \mathbf{v}^c. \quad (\text{G.15})$$

A difference between the Madelung approach and the previous particle-based one is that the colour current is not proportional to the singlet current. Equation (G.14) may be decomposed as

$$\mathbf{J}_a = Q_a \rho \mathbf{v} + \frac{\rho}{2} R_{ab} \mathbf{v}_\perp^b \quad (\text{G.16})$$

where the ‘orthogonal’ velocity \mathbf{v}_\perp^a is defined as

$$\mathbf{v}_\perp^a = (\delta^{ab} - t^a t^b) \mathbf{v}^b. \quad (\text{G.17})$$

Equation (G.15) shows that colour current possesses components that are orthogonal to the singlet current.

In a postscript at the end of this appendix, we derive for the $SU(2)$ case the decomposition of the Schrödinger equation with the parametrization (G.8). Two equations emerge: one regains the conservation of the Abelian current and the other is the ‘Bernoulli’ equation,

$$(g^{-1}\dot{g})^a = \left[\mathbf{v}^b \cdot \mathbf{v}^b - \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right] t^a + \frac{1}{\rho} \nabla \cdot (\rho \epsilon_{abc} \mathbf{v}^b t^c). \quad (\text{G.18})$$

It is further verified that the covariant conservation of the colour current is a consequence of the Abelian continuity equation and (G.18). However, there is no Wong equation because the colour current is not proportional to the conserved singlet current. Finally, using the identity, which follows from the definition (G.12),

$$\dot{\mathbf{v}}^a = -\frac{1}{2} \nabla (g^{-1}\dot{g})^a + \epsilon_{abc} \mathbf{v}^b (g^{-1}\dot{g})^c, \quad (\text{G.19})$$

one can deduce an Euler equation for $\dot{\mathbf{v}}^a$ from (G.18).

We record the energy and momentum density,

$$\mathcal{E} = \frac{1}{2} \nabla \Psi^\dagger \cdot \nabla \Psi = \frac{1}{2} \rho \mathbf{v}^a \cdot \mathbf{v}^a + \frac{\nabla \rho \cdot \nabla \rho}{8\rho} \quad (\text{G.20})$$

$$\mathcal{P} = \frac{i}{2} (\nabla \Psi^\dagger \Psi - \Psi^\dagger \nabla \Psi) = \mathbf{v} \rho. \quad (\text{G.21})$$

Both parallel and orthogonal components of the velocity contribute to the energy density but only the parallel component \mathbf{v} contributes to the momentum density. It is clear that within the present approach the fluid colour flows in every direction in the group space, but the mass density is carried by the unique velocity \mathbf{v} . This is in contrast to our previous approach where all motion is in a single direction or at most in the directions of the Cartan elements of the Lie algebra (section 5.4).

The difference between the two approaches is best seen from a comparison of Lagrangians. For the colour Schrödinger theory in the Madelung representation

$$\mathcal{L}_{\text{Schrödinger}} = \frac{i}{2} (\Psi^\dagger \dot{\Psi} - \dot{\Psi}^\dagger \Psi) - \frac{1}{2} \nabla \Psi^\dagger \cdot \nabla \Psi \quad (\text{G.22})$$

$$= i\rho u^\dagger g^{-1} \dot{g} u - \frac{1}{2} \rho \mathbf{v}^a \cdot \mathbf{v}^a - \frac{(\nabla \rho)^2}{8\rho}. \quad (\text{G.23})$$

With $u \otimes u^\dagger \equiv I/2 - 2iK$, the free part of the above reads

$$\mathcal{L}_{\text{Schrödinger}}^0 = \rho 2 \text{Tr} K g^{-1} \dot{g} - \frac{1}{2} \rho \mathbf{v}^a \cdot \mathbf{v}^a. \quad (\text{G.24})$$

On the other hand, the free part of the Lagrange density (5.28) in the nonrelativistic limit, with $f(n)$ given by (5.45), is

$$\begin{aligned} \mathcal{L}^0 &= \rho 2 \text{Tr}(K g^{-1} \dot{g} + K \mathbf{v} \cdot g^{-1} \nabla g) - \sqrt{\rho^2 (c^2 - \mathbf{v}^2)}, \\ &\approx \rho 2 \text{Tr}(K g^{-1} \dot{g} + K \mathbf{v} \cdot g^{-1} \nabla g) - \rho c^2 + \frac{1}{2} \rho \mathbf{v}^2, \\ &= \rho 2 \text{Tr} K g^{-1} \dot{g} - \frac{1}{2} \rho \mathbf{v}^2 - \rho c^2, \end{aligned} \quad (\text{G.25})$$

where we have used $\mathbf{v} = -2 \text{Tr} K g^{-1} \nabla g$, which follows upon the variation of \mathbf{v} , in the next-to-last equality above. Thus the canonical 1-form is the same for both models while the difference resides in the velocity dependence of their respective Hamiltonians. Only the singlet \mathbf{v} enters (G.25) while the Madelung construction uses the group vector \mathbf{v}^a .

Finally, note that while the Euler equation, which emerges when (G.18) and (G.19) are combined, intricately couples all directions of the fluid velocity \mathbf{v}^a , it does admit the simple solution $\mathbf{v}^a = \mathbf{v} t^a$, with \mathbf{v} obeying the Abelian equations that arise from (G.3)–(G.4).

Postscript. When (G.8) is inserted into (G.6), and use is made of the definition (G.12), we find in the $SU(2)$ case

$$\frac{1}{2} i \dot{\rho} u + i \rho (g^{-1} \dot{g})^a T^a u = -\frac{1}{2} \sqrt{\rho} \nabla^2 \sqrt{\rho} u + \nabla(\rho \mathbf{v}^a) T^a u + \frac{1}{2} \rho \mathbf{v}^a \cdot \mathbf{v}^a u. \quad (\text{G.26})$$

Next (G.26) is premultiplied by u^\dagger , where it implies

$$i \dot{\rho} + \rho (g^{-1} \dot{g})^a t^a = -\sqrt{\rho} \nabla^2 \sqrt{\rho} - i \nabla(\rho \mathbf{v}^a t^a) + \rho \mathbf{v}^a \cdot \mathbf{v}^a. \quad (\text{G.27})$$

The imaginary part reproduces the continuity equation for the singlet current, while the real part gives

$$(g^{-1} \dot{g})^a T^a = -\frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} + \mathbf{v}^a \cdot \mathbf{v}^a. \quad (\text{G.28})$$

To obtain further information, we premultiply (G.26) with $u^\dagger T^b$. This gives

$$\begin{aligned} \dot{\rho} t^b - i \rho (g^{-1} \dot{g})^b + \rho (g^{-1} \dot{g})^a \epsilon_{bac} T^c \\ = i \sqrt{\rho} \nabla^2 \sqrt{\rho} T^b - \nabla \cdot (\rho \mathbf{v}^b) - i \nabla \cdot (\rho \mathbf{v}^a) \epsilon_{bac} T^c - i \rho \mathbf{v}^a \cdot \mathbf{v}^a T^b. \end{aligned} \quad (\text{G.29})$$

The imaginary part gives (G.18) while the real part is identically satisfied by virtue of the Abelian continuity equation and (G.18).

References

- [1] Schwinger J 1961 *J. Math. Phys.* **2** 407
 Bakshi P M and Mahanthappa K 1963 *J. Math. Phys.* **4** 12
 Keldysh L V 1964 *Zh. Eksp. Teor. Fiz.* **47** 151
 Keldysh L V 1964 *Sov. Phys.—JETP* **20** 1018 (Engl. Transl.)
- [2] A standard physics text on the subject is L Landau and E Lifshitz 1987 *Fluid Mechanics* 2nd edn (Oxford: Pergamon)
 A mathematical treatment is V Arnold and B Khesin 1988 *Topological Methods in Hydrodynamics* (Berlin: Springer)
 The relation between Lagrange and Euler descriptions of a fluid is discussed by Salmon R 1988 *Ann. Rev. Fluid Mech.* **20** 225
 see also Antoniou I and Pronko G *Preprint* hep-th/0106119

- [3] These brackets were first posited (in somewhat imprecise form) Landau L 1941 *Zh. Eksp. Teor. Fiz.* **11** 592
Landau L 1941 *J. Phys. USSR* **5** 71 (Engl. Transl.)
see also Morrison P and Greene J 1980 *Phys. Rev. Lett.* **45** 790
Morrison P and Greene J 1982 *Phys. Rev. Lett.* **E 48** 569
- [4] In particle dynamics the inverse square potential is $SO(2,1)$ invariant R. Jackiw 1972 *Phys. Today* **25** 23
Niederer U 1972 *Helv. Phys. Acta* **45** 802
Hagen C R 1972 *Phys. Rev. D* **5** 377
In fluid mechanics the equation of state that follows from (1.57b): pressure $\propto \rho^{1+2/d}$ enjoys Schrödinger group invariance
See Hassaine M and Horvathy P 2000 *Ann. Phys., NY* **282** 218
O’Raifeartaigh L and Sreedhar V 2001 *Ann. Phys., NY* **293** 215
- [5] Chaplygin S 1904 *Sci. Mem. Mosc. Univ. Math. Phys.* **21** 1
Chaplygin was a colleague of fellow USSR Academician N Luzin 1997 Although accused by Stalinist authorities of succumbing excessively to foreign influences, unaccountably both managed to escape the fatal consequences of their alleged actions; see N Kremetsov *Stalinist Science* (Princeton, NJ: Princeton University Press)
The same model (1.60) was later put forward by Tsien H-S 1939 *J. Aeronaut. Sci.* **6** 399
von Karman T 1941 *J. Aeronaut. Sci.* **8** 337
see also Stanyukovich K 1960 *Unsteady Motion of Continuous Media* (Oxford: Pergamon) p 128
- [6] For an introduction to the properties of the Abelian and non-Abelian Chern–Simons terms see Deser S, Jackiw R and Templeton S 1982 *Ann. Phys., NY* **140** 372
For an introduction to the properties of the Abelian and non-Abelian Chern–Simons terms, see Deser S, Jackiw R and Templeton S 1985 *Ann. Phys., NY* **E 185** 406
In fluid mechanics or in magnetohydrodynamics the Abelian Chern–Simons term is known as the fluid or magnetic helicity, and was introduced by L Woltier 1958 *Proc. Natl. Acad. Sci.* **44** 489 and further studied in Berger M and Field G 1984 *J. Fluid Mech.* **147** 133
Moffatt H and Tsinober A 1992 *Ann. Rev. Fluid Mech.* **24** 281
- [7] Faddeev L and Jackiw R 1988 *Phys. Rev. Lett.* **60** 1692
For a detailed exposition, see Jackiw R 1994 *Constraint Theory and Quantization Methods* ed F Colomo, L Lusanna and G Marmo (Singapore: World Scientific)
Jackiw R 1995 *Diverse Topics in Theoretical and Mathematical Physics* (Singapore: World Scientific) (reprint)
- [8] Bak D, Jackiw R and Pi S-Y 1994 *Phys. Rev. D* **49** 6678 (appendix)
- [9] See for example Arnold V 1989 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
Guillemin V and Sternberg S 1990 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press)
- [10] Lin C C 1963 *International School of Physics E Fermi (XXI)* ed G Careri (New York: Academic)
- [11] Clebsch A 1859 *J. Reine Angew. Math.* **56** 1
- [12] Eckart C 1938 *Phys. Rev.* **54** 920
- [13] Lamb H 1932 *Hydrodynamics* (Cambridge: Cambridge University Press) p 248
- [14] Deser S, Jackiw R and Polychronakos A P 2001 *Phys. Lett. A* **279** 151
- [15] Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley)
- [16] Bazeia D and Jackiw R 1998 *Ann. Phys., NY* **270** 246
- [17] Bordemann M and Hoppe J 1993 *Phys. Lett. B* **317** 315
Jevicki A 1998 *Phys. Rev. D* **57** 5955
- [18] Bazeia D 1999 *Phys. Rev. D* **59** 085007
- [19] Ogawa N 2000 *Phys. Rev. D* **62** 085023
- [20] Jackiw R and Polychronakos A P 1999 *Commun. Math. Phys.* **207** 107
- [21] Bordemann M and Hoppe J 1994 *Phys. Lett. B* **325** 359
- [22] The method described in sections 2.55 and 2.56 was presented for $d = 2$ in Hoppe J 1994 *Phys. Rev. B* **329** 10 while a version of the argument in section 2.57, specialized to $d = 2$, is found in [17] Generalization to arbitrary d is given in Jackiw R and Polychronakos A P 1999 *Proc. Steklov Inst. Math.* **226** 193 and [20]
- [23] Susskind L 1968 *Phys. Rev.* **165** 1535
- [24] Nutku Y 1987 *J. Math. Phys.* **28** 2579
Olver P and Nutku Y 1988 *J. Math. Phys.* **29** 1610
Arik M, Neyzi F, Nutku Y, Olver P and Verosky J 1989 *J. Math. Phys.* **30** 1338
Brunelli J and Das A 1997 *Phys. Lett. A* **235** 597
- [25] Barbishov B and Chernikov N 1966 *Zh. Eksp. Teor. Fiz.* **51** 658
Barbishov B and Chernikov N 1967 *Sov. Phys.—JETP* **24** 437 (Engl. Transl.)
- [26] Grundland A, Hamilton A and Hussin V 2003 *J. Math. Phys.* **44** 2874

- [27] Jevicki A and Sakita B 1980 *Phys. Rev. D* **22** 467
Jevicki A and Sakita B 1980 *Nucl. Phys. B* **165** 511
Polchinski J 1991 *Nucl. Phys. B* **362** 25
Avan J and Jevicki A 1991 *Phys. Lett. B* **266** 35
Avan J and Jevicki A 1991 *Phys. Lett. B* **272** 17
- [28] Floreanini R and Jackiw R 1987 *Phys. Rev. Lett.* **59** 1873
- [29] Jackiw R and Polychronakos A P 2000 *Phys. Rev. D* **62** 085019
some of these results are described in unpublished papers by Hoppe J 1993 *Karlsruhe Preprints* KA-THEP-6-93, KA-THEP-9-93, hep-th/9311059
- [30] Bergner Y and Jackiw R 2001 *Phys. Lett. A* **284** 146
- [31] Duff M J 1997 *Fields, Strings and Duality* ed C Efthimiou and B Greene (Singapore: World Scientific)
- [32] de Wit B, Hoppe J and Nicolai H 1988 *Nucl. Phys. B* **305** FS23 525
- [33] Hassaine M 2001 *Phys. Lett. A* **290** 157
Nyawelo T, Groot Nibbelink S and van Holten J 2001 *Phys. Rev. D* **64** 021701
Nyawelo T, Groot Nibbelink S and van Holten J 2003 *Phys. Rev. D* **68** 125006
Das A and Popowicz Z 2002 *Phys. Lett. A* **296** 15
Nyawelo T 2003 *Nucl. Phys. B* **672** 87
- [34] See, for example, Ludlam T and McLerran L 2003 *Phys. Today* **56** 48
- [35] Pisarski R 1989 *Physica A* **158** 246
Pisarski R 1989 *Phys. Rev. Lett.* **63** 1129
Braaten E and Pisarski R 1990 *Phys. Rev. D* **42** 2156
for a recent update and later references, see Kraemer U and Rebhan A 2004 *Rep. Prog. Phys.* **67** 351
- [36] Linde A D 1980 *Phys. Lett. B* **96** 289
Gross D, Pisarski R and Yaffe L 1981 *Rev. Mod. Phys.* **53** 43
- [37] Heinz U 1985 *Ann. Phys., NY* **161** 48
Elze H-T and Heinz U 1989 *Phys. Rep.* **183** 81
for recent update and later references see Litin D and Manuel C 2002 *Phys. Rep.* **364** 451
- [38] Wong S K 1970 *Nuovo Cimento A* **65** 689
- [39] Bistrovic B, Jackiw R, Li H, Nair V P and Pi S-Y 2003 *Phys. Rev. D* **67** 025013
- [40] Jackiw R and Johnson K 1969 *Phys. Rev.* **182** 1459
Gross D and Jackiw R 1970 *Nucl. Phys. B* **14** 269
- [41] Balachandran A P, Borchardt S and Stern A 1978 *Phys. Rev. D* **17** 3247
Balachandran A, Marmo G, Skagerstam B-S and Stern A 1982 *Gauge Symmetries and Fibre Bundles (Lecture Notes in Physics vol 188)* (Berlin: Springer)
- [42] Madelung E 1927 *Z. Phys.* **40** 322
see also Merzbacher E 1998 *Quantum Mechanics* 3rd edn (New York: Wiley)
- [43] Similar investigations are by Bohm D, Schiller R and Tiomno J 1995 *Nuovo Cimento Suppl.* **1** 48
Love P and Boghosian B 2002 *Physica A* **332** 47
- [44] Jackiw R, Pi S-Y and Polychronakos A 2002 *Ann. Phys., NY* **301** 157
- [45] Susskind L 2001 *Preprint* hep-th/0101029
- [46] Seiberg N and Witten E 1999 *J. High Energy Phys.* JHEP09(1999)032
- [47] Okawa Y and Ooguri H 2001 *Phys. Lett. D* **64** 046009
- [48] Jackiw R and Pi S-Y 2002 *Phys. Rev. Lett.* **88** 111603
- [49] Jackiw R 1978 *Phys. Rev. Lett.* **41** 1635
Jackiw R 1980 *Acta Phys. Aust. Suppl.* **22** 383
Jackiw R 1995 *Diverse Topics in Theoretical and Mathematical Physics* (Singapore: World Scientific) (reprint)
- [50] Guralnik Z, Jackiw R, Pi S-Y and Polychronakos A 2001 *Phys. Lett. B* **517** 450
Cai R 2001 *Phys. Lett. B* **517** 457
Zahn J 2004 *Preprint* hep-th/0405253
- [51] Madore J, Schraml S, Schupp P and Wess J 2000 *Eur. Phys. J. C* **16** 161
Jurco B, Schraml S, Schupp P and Wess J 2000 *Eur. Phys. J. C* **17** 521
Gross D J, Hashimoto A and Itzhaki N 2000 *Adv. Theor. Math. Phys.* **4** 893
Bak D, Lee K and Park J-H 2001 *Phys. Lett. B* **501** 305
- [52] Polychronakos A P 2002 *Ann. Phys., NY* **174**
- [53] Feynman R P 1972 *Statistical Mechanics* (Reading, MA: Benjamin) p 319
- [54] Dashen R and Sharp D 1968 *Phys. Rev.* **165** 1857
Grodnik J and Sharp D 1970 *Phys. Rev. D* **6** 1531, 1546
Goldin G 1971 *J. Math. Phys.* **12** 462

-
- [55] Dunne G, Jackiw R and Trugenberger C 1990 *Phys. Rev. D* **41** 661
Dunne G and Jackiw R 1993 *Nucl. Phys. B* **33** 114
- [56] Magro G 2003 *Preprint* quant-ph/0302001
- [57] Fairlie D, Fletcher P and Zachos C 1989 *Phys. Lett. B* **218** 203
Fairlie D and Zachos C 1989 *Phys. Lett. B* **224** 101
Duval C, Horvath Z and Horvathy P 2001 *Int. J. Mod. Phys. B* **15** 3397
- [58] Nair V P and Ray R 2004 *Nucl. Phys. B* **676** 659
- [59] Balakrishnan R and Bishop A 1985 *Phys. Rev. Lett.* **55** 537
Haldane F 1986 *Phys. Rev. Lett.* **57** 1488
- [60] Jackiw R and Pi S-Y 2000 *Phys. Rev. D* **61** 105015
- [61] Treiman S, Jackiw R, Zumino B and Witten E 1985 *Current Algebra and Anomalies* (Princeton, NJ/Singapore: Princeton University Press/World Scientific)
- [62] Jackiw R, Nair V P and Pi S-Y 2000 *Phys. Rev. D* **62** 085018
- [63] Bott R and Chern S S 1965 *Acta Math.* **114** 71